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HECKE OPERATORS ON HILBERT–SIEGEL MODULAR FORMS

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We define Hilbert–Siegel modular forms and Hecke "operators" acting on them. As with Hilbert modular forms (i.e. with Siegel degree 1), these linear transformations are not linear operators until we consider a direct product of spaces of modular forms (with varying groups), modulo natural identifications we can make between certain spaces. With Hilbert–Siegel forms (i.e. with arbitrary Siegel degree) we identify several families of natural identifications between certain spaces of modular forms. We associate the Fourier coefficients of a form in our product space to even integral lattices, independent of basis and choice of coefficient rings. We then determine the action of the Hecke operators on these Fourier coefficients, paralleling the result of Hafner and Walling for Siegel modular forms (where the number field is the field of rationals).

Keywords: Siegel modular forms; Hilbert modular forms; Hecke operators.

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1. Introduction

A Siegel modular form F of degree n over the rationals has a Fourier series supported on even integral symmetric $n \times n$ matrices. An even integral symmetric matrix can be interpreted as the matrix for a quadratic form on an even integral lattice, relative to some $\mathbb{Z}$-basis for that lattice. Given the transformation property of F under the symplectic group, the coefficient of F attached to a matrix T is equal to that attached to the conjugate ${}^tGTG$ where G is any integral change of basis matrix (with determinant 1 when k, the weight of the modular form, is odd). Consequently we can rewrite F as a "Fourier series" supported on even integral lattices, without specifying a basis for each lattice. For each prime p there are $n+1$ Hecke operators, $T(p)$ and $T_j(p^2)$ $(1 \le j \le n)$ associated to p, n of which are algebraically independent. In [5] we determined the action of these operators on the Fourier coefficients of F. In this paper we extend this result to Hilbert–Siegel modular forms.

With $\mathbb{K}$ a totally real number field and $\mathcal{P}$ a prime ideal, we mimic the construction of the classical Hecke operators and construct a linear transformation $T(\mathcal{P})$ acting on Hilbert modular forms. When $\mathcal{P}$ is not principally generated, $T(\mathcal{P})$ maps modular forms attached to $\Gamma = SL_2(\mathcal{O})$ ($\mathcal{O}$ the ring of integers of $\mathbb{K}$), to forms attached to the "psuedo-conjugate"

$$\begin{pmatrix} \mathcal{P} & 0 \\ 0 & 1 \end{pmatrix} \Gamma \begin{pmatrix} \mathcal{P}^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} :\ a, d \in \mathcal{O},\ b \in \mathcal{P},\ c \in \mathcal{P}^{-1},\ ad - bc = 1 \right\}.$$

So for $T(\mathcal{P})$ to be a linear operator (meaning its domain and codomain are equal), it is necessary to consider a (finite) direct product of spaces of modular forms attached to psuedo-conjugates of $SL_2(\mathcal{O})$. In [7], Shimura defined "Fourier coefficients" attached to integral ideals of a form in this direct product, and he determined the action of $T(\mathcal{P})$ on these Fourier coefficients.

In the case of Hilbert–Siegel modular forms, we need to consider a (finite) direct product of spaces of forms attached to psuedo-conjugates of $Sp_n(\mathcal{O})$ for the maps $T(\mathcal{P})$ and $T_j(\mathcal{P}^2)$ to be linear operators. For a form in this direct product, we define "Fourier coefficients" attached to even integral lattices, independent of basis and choices of coefficient rings (note that an $\mathcal{O}$-lattice is not necessarily free, and there are numerous ways to write it as $\mathcal{A}_1 x_1 \oplus \cdots \oplus \mathcal{A}_n x_n$ with the $\mathcal{A}_i$ fractional ideals). Then we determine the action of the Hecke operators on these Fourier coefficients. When k is odd, we need to impose an orientation on Λ. Thus

$$F(\tau) = \sum_{\mathrm{cls}\Lambda} c(\Lambda) e^*\{\Lambda\tau\}$$

where $\mathrm{cls}\Lambda$ runs over isometry classes of lattices Λ, and $e^*\{\Lambda\tau\} = \sum_G \exp(\pi i \sigma Tr({}^tGTG\tau))$; here $\Lambda = \mathcal{A}_1 x_1 \oplus \cdots \oplus \mathcal{A}_n x_n$, $T = (B(x_i, x_j))$ where B is the symmetric bilinear form associated to the quadratic form Q on Λ so that $Q(x) = B(x, x)$, σ is the trace function on matrices, and G varies over $O(\Lambda)\backslash GL_n(\mathcal{O})$ when k is even, and over $O^+(\Lambda)\backslash SL_n(\mathcal{O})$ when k is odd. (Two lattices Λ, Ω are in the same isometry class if there is an isomorphism from one onto the other that preserves the quadratic form. Also, $O(\Lambda)$ is the orthogonal group of Λ.)

We begin by defining symplectic groups $\Gamma(\mathcal{I}_1, \ldots, \mathcal{I}_n; \mathcal{J})$ for fractional ideals $\mathcal{I}_i, \mathcal{J}$. We show that the spaces of modular forms associated to $\Gamma(\mathcal{I}_1, \ldots, \mathcal{I}_n; \mathcal{J})$ and $\Gamma(\mathcal{I}'_1, \ldots, \mathcal{I}'_n; \mathcal{J}')$ are naturally isomorphic whenever $\mathrm{cls}(\mathcal{I}_1 \cdots \mathcal{I}_n) = \mathrm{cls}(\mathcal{I}'_1 \cdots \mathcal{I}'_n)$ and $\mathrm{clx}^+ \mathcal{J} = \mathrm{clx}^+ \mathcal{J}'$. (Here $\mathrm{cls}\,\mathcal{I}$ denotes the wide ideal class of $\mathcal{I}$, and $\mathrm{clx}^+ \mathcal{J}$ denotes the strict ideal complex of $\mathcal{J}$. Thus $\mathrm{cls}\,\mathcal{I} = \mathrm{cls}\,\mathcal{I}'$ if $\mathcal{I} = \alpha\mathcal{I}'$ for some $\alpha \in \mathbb{K}$, and $\mathrm{clx}^+ \mathcal{J} = \mathrm{clx}^+ \mathcal{J}'$ if $\mathcal{J} = \alpha \mathcal{I}^2 \mathcal{J}'$ for some fractional ideal $\mathcal{I}$ and $\alpha \gg 0$.) We set $\mathcal{M}_k = \otimes_{\mathcal{I}_i, \mathcal{J}} \mathcal{M}_k(\Gamma(\mathcal{I}_1, \ldots, \mathcal{I}_n; \mathcal{J}))/\sim$ (so we identify spaces that are naturally isomorphic). Next we attach the Fourier coefficients of (the components of) F to even integral lattices, independent of the basis and the coefficient rings used to realize each lattice. (For a full discussion of this, see the discussion preceding

Proposition 2.2.) In Sec. 3 we introduce operators $S(\mathcal{Q})$ attached to fractional ideals $\mathcal{Q}$, and we decompose $\mathcal{M}_k$ as $\oplus_\chi \mathcal{M}_k(\chi)$ where χ varies over ideal class characters, and $F|S(\mathcal{Q}) = \chi(\mathcal{Q})F$ for $F \in \mathcal{M}_k(\chi)$. Then in Sec. 4 we introduce the Hecke operators $T(\mathcal{P})$ and $T_j(\mathcal{P}^2)$, $0 \le j \le n$, and we find coset representatives giving the action of the operators. When then analyzing the action of the Hecke operators $T_j(\mathcal{P}^2)$ in Sec. 5 we encounter incomplete character sums; we complete these by replacing $T_j(\mathcal{P}^2)$ with $\widetilde{T}_j(\mathcal{P}^2)$, a combination of $T_\ell(\mathcal{P}^2)$, $0 \le \ell \le j$. Finally, we show that for $\Lambda^{\mathcal{J}}$ an even integral lattice and $F \in \mathcal{M}_k(\chi)$, the $\Lambda^{\mathcal{J}}$th coefficient of $F|\widetilde{T}_j(\mathcal{P}^2)$ is

$$\sum_{\mathcal{P}\Lambda \subseteq \Omega \subseteq \mathcal{P}^{-1}\Lambda} N(\mathcal{P})^{E_j(\Omega,\Lambda)} \chi(\mathcal{P})^{e_j(\Omega,\Lambda)} \alpha_j(\Omega,\Lambda) c_F(\Omega^{\mathcal{J}})$$

where $E_j(\Omega,\Lambda)$ and $e_j(\Omega,\Lambda)$ are given by formulas in terms of the invariant factors $\{\Omega : \Lambda\}$, and $\alpha_j(\Omega,\Lambda)$ reflects some geometry of $(\Omega \cap \Lambda)/\mathcal{P}(\Omega + \Lambda)$. (A formula for $\alpha_j(\Omega,\Lambda)$ is given at the end of Sec. 5.) A similar but much simpler argument shows that the $\Lambda^{\mathcal{J}}$th coefficient of $F|T(\mathcal{P})$ is

$$\sum_{\mathcal{P}\Lambda \subseteq \Omega \subseteq \Lambda} N(\mathcal{P})^{E(\Omega,\Lambda)} c_F(\Omega^{\mathcal{J}\mathcal{P}^{-1}})$$

(see Theorem 5.2).

In Sec. 6 we present a lemma on completing a symmetric coprime pair to a symplectic matrix. The reader is referred to [6] for basic results on lattices and quadratic forms.

2. Definitions, Isomorphisms, and Fourier Coefficients Attached to Even Integral Lattices

Let $\mathbb{K}$ be a totally real number field of degree d over $\mathbb{Q}$, and let ∂ denote the different of $\mathbb{K}$. Let $\mathcal{H}_{(n)}$ denote degree n Siegel upper half-space; so

$$\mathcal{H}_{(n)} = \{\tau = X + iY : \ X, Y \in \mathbb{R}^{n,n} \text{ are symmetric, } Y > 0\}.$$

For fractional ideals $\mathcal{I}_1, \ldots, \mathcal{I}_n, \mathcal{J}$, let

$$\begin{aligned}\Gamma(\mathcal{I}_1, \ldots, \mathcal{I}_n; \mathcal{J}) = \Bigg\{ &\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G_{2n}(\mathbb{K}) : \ A^tB, C^tD \text{ symmetric,} \\ &A^tD - B^tC = uI, \ u \in \mathcal{O}^+, \ a_{ij} \in \mathcal{I}_i\mathcal{I}_j^{-1}, \\ &b_{ij} \in \mathcal{I}_i\mathcal{I}_j\mathcal{J}\partial^{-1}, \ c_{ij} \in (\mathcal{I}_i\mathcal{I}_j\mathcal{J})^{-1}\partial, \ d_{ij} \in \mathcal{I}_i^{-1}\mathcal{I}_j \Bigg\}.\end{aligned}$$

(Here $A = (a_{ij})$, etc.) So with $\Gamma = \Gamma(\mathcal{O}, \dots, \mathcal{O}; \mathcal{O})$, $\Gamma(\mathcal{I}_1, \dots, \mathcal{I}_n; \mathcal{J})$ corresponds to the formal conjugate $\underline{\delta}\Gamma\underline{\delta}^{-1}$ where

$$\underline{\delta} = \begin{pmatrix} \mathcal{J}\mathcal{I}_1 & & & & & \\ & \ddots & & & & \\ & & \mathcal{J}\mathcal{I}_n & & & \\ & & & \mathcal{I}_1^{-1} & & \\ & & & & \ddots & \\ & & & & & \mathcal{I}_n^{-1} \end{pmatrix}.$$

Also notice that $\Gamma(\mathcal{I}_1, \dots, \mathcal{I}_n; \mathcal{J}\mathcal{I}^2) = \Gamma(\mathcal{I}_1\mathcal{I}, \dots, \mathcal{I}_n\mathcal{I}; \mathcal{J})$.

Definition. A degree n ($n > 1$), weight k Hilbert–Siegel modular form for $\Gamma(\mathcal{I}_1, \dots, \mathcal{I}_n; \mathcal{J})$ is a function $f : \mathcal{H}_{(n)}^d \to \mathbb{C}$ so that the following two conditions hold:

(1) f is "analytic on $\mathcal{H}_{(n)}^d$ and at infinity," meaning that for $\tau \in \mathcal{H}_{(n)}^d$,

$$f(\tau) = \sum_T c(T)e\{T\tau\}$$

where T runs over symmetric, positive semi-definite $n \times n$ matrices. Also, $\sigma(M)$ denotes the trace of a matrix M, Tr denotes the trace from $\mathbb{K}$ to $\mathcal{Q}$, and

$$e\{T\tau\} = \exp(\pi i \sigma(\mathrm{Tr}(T\tau))).$$

Here $\mathrm{Tr}(T\tau) = \sum_{i=1}^d T^{(i)}\tau_i$, where $T^{(i)}$ is the image of T under the ith embedding of $\mathbb{K}$ into $\mathbb{R}$.

(2) For all $M \in \Gamma(\mathcal{I}_1, \dots, \mathcal{I}_n; \mathcal{J})$, $f|M = f$ where, for any matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ (written in $n \times n$ blocks), we define

$$f \left| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right. (\tau) = \det(N(A^tD - B^tC))^{k/2} \det(N(C\tau + D))^{-k} \times f((A\tau + B)(C\tau + D)^{-1}).$$

Here N denotes the norm from $\mathbb{K}$ to $\mathbb{Q}$, extended so that

$$N(C\tau + D) = \prod_{i=1}^d C^{(i)}\tau_i + D^{(i)}.$$

Let $\mathcal{M}_k(\Gamma(\mathcal{I}_1, \dots, \mathcal{I}_n; \mathcal{J}))$ denote the space of Hilbert–Siegel modular forms for $\Gamma(\mathcal{I}_1, \dots, \mathcal{I}_n; \mathcal{J})$, and let f be a modular form in this space. Since $f(\tau + B) = f(\tau)$ for all symmetric $B \in (\mathcal{I}_i\mathcal{I}_j\mathcal{J}\partial^{-1})$, ∂ the different of $\mathbb{K}$, we must have $e\{TB\} = 1$ for all $T \in \operatorname{supp} f$. Note that for $T = (t_{ij})$, $B = (b_{ij})$ symmetric matrices,

$$\sigma(TB) = \sum_{i=1}^n t_{ii}b_{ii} + \sum_{1 \le i < j \le n} 2t_{ij}b_{ij}.$$

Thus for $T \in \operatorname{supp} f$, we must have $T \in ((I_i I_j \mathcal{J})^{-1})$ with T even, meaning $t_{ii} \in 2I_i^{-2}\mathcal{J}^{-1}$.

Definitions. We define families of isomorphisms between spaces of modular forms as follows: Fix $f \in \mathcal{M}_k(\Gamma(\mathcal{I}_1, \ldots, \mathcal{I}_n; \mathcal{J}))$.

First, for $\alpha \in \mathbb{K}^\times$ and $1 \le \ell \le n$, let

$$M = \begin{pmatrix} I_{\ell-1} & & \\ & \alpha^{-1} & \\ & & I_{n-\ell} \end{pmatrix},$$

and define

$$f|U_\ell(\alpha) = f \left| \begin{pmatrix} M & \\ & {}^t M^{-1} \end{pmatrix} \right. .$$

Since $\left(\begin{smallmatrix} M & \\ & {}^tM^{-1} \end{smallmatrix}\right)\Gamma(\mathcal{I}'_1, \ldots, \mathcal{I}'_n; \mathcal{J})\left(\begin{smallmatrix} M^{-1} & \\ & {}^tM \end{smallmatrix}\right) = \Gamma(\mathcal{I}_1, \ldots, \mathcal{I}_n; \mathcal{J})$ where

$$\mathcal{I}'_i = \begin{cases} \mathcal{I}_i & \text{if } \ell \ne i \\ \alpha\mathcal{I}_\ell & \text{if } \ell = i, \end{cases}$$

$U_\ell(\alpha)$ defines an isomorphism from $\mathcal{M}_k(\Gamma(\mathcal{I}_1, \ldots, \mathcal{I}_n; \mathcal{J}))$ onto $\mathcal{M}_k(\Gamma(\mathcal{I}'_1, \ldots, \mathcal{I}'_n; \mathcal{J}))$.

For $\alpha \gg 0$, define $W(\alpha) : \mathcal{M}_k(\Gamma(\mathcal{I}_1, \ldots, \mathcal{I}_n; \mathcal{J})) \to \mathcal{M}_k(\Gamma(\mathcal{I}_1, \ldots, \mathcal{I}_n; \alpha\mathcal{J}))$ by

$$f|W(\alpha) = f \left| \begin{pmatrix} \alpha^{-1} I_n & \\ & I_n \end{pmatrix} \right. .$$

One easily checks (as we did above for $U_\ell(\alpha)$) that $W(\alpha)$ is an isomorphism.

For $\mathcal{Q}$ a fractional ideal, $1 \le \ell < n$, choose

$$A \in \begin{pmatrix} \mathcal{Q}^{-1} & \mathcal{Q}\mathcal{I}_\ell\mathcal{I}_{\ell+1}^{-1} \\ \mathcal{Q}^{-1}\mathcal{I}_\ell^{-1}\mathcal{I}_{\ell+1} & \mathcal{Q} \end{pmatrix}$$

so that $\det A = 1$ (possible by Strong Approximation; see [6, p. 42]). Let

$$M = \begin{pmatrix} I_{\ell-1} & & \\ & A & \\ & & I_{n-\ell-1} \end{pmatrix},$$

and define

$$f|V_\ell(\mathcal{Q}) = f \left| \begin{pmatrix} M & \\ & {}^t M^{-1} \end{pmatrix} \right. .$$

Since $\left(\begin{smallmatrix} M & \\ & {}^tM^{-1} \end{smallmatrix}\right)\Gamma(\mathcal{I}'_1, \ldots, \mathcal{I}'_n; \mathcal{J})\left(\begin{smallmatrix} M^{-1} & \\ & {}^tM \end{smallmatrix}\right) = \Gamma(\mathcal{I}_1, \ldots, \mathcal{I}_n; \mathcal{J})$

where

$$\mathcal{I}'_i = \begin{cases} \mathcal{I}_i & \text{if } i \ne \ell, \quad \ell+1 \\ \mathcal{Q}\mathcal{I}_\ell & \text{if } i = \ell \\ \mathcal{Q}^{-1}\mathcal{I}_{\ell+1} & \text{if } i = \ell+1, \end{cases}$$

the map $V_\ell(\alpha)$ defines an isomorphism from $\mathcal{M}_k(\Gamma(\mathcal{I}_1,\dots,\mathcal{I}_n;\mathcal{J}))$ onto $\mathcal{M}_k(\Gamma(\mathcal{I}'_1,\dots,\mathcal{I}'_n;\mathcal{J}))$.

For $\mathcal{Q}$ a fractional ideal and $1\le \ell<j\le n$, set

$$V_{\ell j}(\mathcal{Q}) = V_\ell(\mathcal{Q})V_{\ell+1}(\mathcal{Q})\cdots V_{j-1}(\mathcal{Q}).$$

Then $V_{\ell j}(\mathcal{Q})$ defines an isomorphism from $\mathcal{M}_k(\mathcal{I}_1,\dots,\mathcal{I}_n;\mathcal{J}))$ onto $\mathcal{M}_k(\Gamma(\mathcal{I}'_1,\dots,\mathcal{I}'_n;\mathcal{J}))$, where

$$\mathcal{I}'_i = \begin{cases} \mathcal{I}_i & \text{if } i\ne \ell,j \\ \mathcal{Q}\mathcal{I}_\ell & \text{if } i=\ell \\ \mathcal{Q}^{-1}\mathcal{I}_{j+1} & \text{if } i=j. \end{cases}$$

Proposition 2.1. *The maps $U_\ell(\alpha)$, $W(\alpha)$, $V_{\ell j}(\mathcal{Q})$ commute and for fixed ℓ,j, these operators are multiplicative* (*as functions on fractional ideals*).

Proof. The tedious aspect of proving such relations among our isomorphisms is that, for any of the above listed maps, the domain and codomain differ. Keeping track of appropriate domains and codomains, and using the matrices we used to define the actions of these operators, it is then straightforward to verify the operators commute, remembering that if $MN^{-1}\in\Gamma$ for any group $\Gamma=\Gamma(\mathcal{I}_1,\dots,\mathcal{I}_n;\mathcal{J})$, then $f|M=f|N$ for $f\in\mathcal{M}_k(\Gamma)$. $\square$

Definition. For $f\in\mathcal{M}_k(\Gamma(\mathcal{I}_1,\dots,\mathcal{I}_n;\mathcal{J}))$, $g\in\mathcal{M}_k(\mathcal{I}'_1,\dots,\mathcal{I}'_n;\mathcal{J}'))$, define the equivalence relation $\sim$ by $f\sim g$ if some composition of the maps U_i, W, V_{ij} takes f to g. We define

$$\mathcal{M}_k = \otimes_{\mathcal{I}_i,\mathcal{J}}\mathcal{M}_k(\Gamma(\mathcal{I}_1,\dots,\mathcal{I}_n;\mathcal{J}))/\sim$$

where $\mathcal{I}_1,\dots,\mathcal{I}_n,\mathcal{J}$ vary over all fractional ideals. Note that $\sim$ partitions the spaces $\mathcal{M}_k(\Gamma(\mathcal{I}_1,\dots,\mathcal{I}_n;\mathcal{J}))$ according to $\mathrm{cls}(\mathcal{I}_1\cdots\mathcal{I}_n)$, $\mathrm{clx}^+\mathcal{J}$. Thus $\mathcal{M}_k\approx\otimes_{\mathrm{cls}\,\mathcal{I},\mathrm{clx}^+\mathcal{J}}\mathcal{M}_k(\Gamma(\mathcal{O},\dots,\mathcal{O},\mathcal{I};\mathcal{J}))$, $\mathrm{cls}\,\mathcal{I}$ runs over all ideal classes and $\mathrm{cls}^+\mathcal{J}$ runs over all strict ideal class complexes. ($\mathcal{J},\mathcal{J}'$ are in the same strict ideal class complex if $\mathcal{J}'=\alpha\mathcal{I}^2\mathcal{J}$ for some fractional ideal $\mathcal{I}$ and $\alpha\gg 0$.)

Let $\mathcal{I}_1,\dots,\mathcal{I}_h$ represent the ideal classes, $\mathcal{J}_1,\dots,\mathcal{J}_m$ the strict ideal complexes. Then for $F\in\mathcal{M}_k$, F is represented by any $(\dots,f_{ij},\dots)$ where $f_{ij}\in\mathcal{M}_k(\Gamma(\mathcal{I}'_1,\dots,\mathcal{I}'_n;\mathcal{J}'))$, $\mathcal{I}'_1\cdots\mathcal{I}'_n\in\mathrm{cls}\,\mathcal{I}_i$, $\mathcal{J}'\in\mathrm{clx}^+\mathcal{J}_j$.

Given an element $F\in\mathcal{M}_k$, we can associate the Fourier coefficients of F with lattices equipped with positive semi-definite, even integral quadratic forms as described below.

First note the following. Say $f\in\mathcal{M}_k(\Gamma(\mathcal{I}_1,\dots,\mathcal{I}_n;\mathcal{J}))$ is a component of a chosen representative for F. So the support of f lies in $\big((\mathcal{I}_i\mathcal{I}_j\mathcal{J})^{-1}\big)$. For $T\in\mathrm{supp}\,f$, consider T as defining a quadratic form Q on $\mathcal{O}x_1\oplus\cdots\oplus\mathcal{O}x_n$ (recall that T is symmetric and even, meaning the ith diagonal entry of T lies in $2\mathcal{I}_i^{-2}\mathcal{J}^{-1}$). Thus with $\Lambda=\mathcal{I}_1x_1\oplus\cdots\oplus\mathcal{I}_nx_n$, $Q(\Lambda)\subseteq 2\mathcal{J}^{-1}$. If $\alpha\mathcal{O}=\mathcal{J}$ then the scaled lattice

Λ^α is even integral, meaning $\alpha Q(\Lambda) \subseteq 2\mathcal{O}$. When $\mathcal{J}$ is not principal, we abuse notation and language and refer to $\Lambda^{\mathcal{J}}$ as an even integral lattice. We agree to identify $\Lambda^{\mathcal{I}^2}$ with $\mathcal{I}\Lambda$ since locally everywhere these are identified (paralleling the fact that $\Gamma(\mathcal{I}_1, \dots, \mathcal{I}_n; \mathcal{I}^2\mathcal{J}) = \Gamma(\mathcal{I}\mathcal{I}_1, \dots, \mathcal{I}\mathcal{I}_n; \mathcal{J})$).

Not every lattice is a free $\mathcal{O}$-module, and given a lattice Λ, there are many ways to choose fractional ideals $\mathcal{I}_j$ and vectors x_j so that $\Lambda = \mathcal{I}_1 x_1 \oplus \cdots \oplus \mathcal{I}_n x_n$. Say we have

$$\Lambda = \mathcal{I}_1 x_1 \oplus \cdots \oplus \mathcal{I}_n x_n = \mathcal{I}'_1 y_1 \oplus \cdots \oplus \mathcal{I}'_n y_n,$$

$y_j = \sum_i a_{ij} x_i$. Then by [6, 81:8], $\mathcal{I}_1 \cdots \mathcal{I}_n = \mathcal{I}'_1 \cdots \mathcal{I}'_n \cdot \det(a_{ij})$.

The Invariant Factor Theorem [6, 81:11] says that given lattices Λ, Ω on a (non-zero) space V, there are vectors $x_1, \dots, x_n \in V$ and fractional ideals $\mathcal{I}_1, \dots, \mathcal{I}_n, \mathcal{A}_1, \dots, \mathcal{A}_n$ so that

$$\begin{aligned} \Lambda &= \mathcal{I}_1 x_1 \oplus \cdots \oplus \mathcal{I}_n x_n, \\ \Omega &= \mathcal{I}_1 \mathcal{A}_1 x_1 \oplus \cdots \oplus \mathcal{I}_n \mathcal{A}_n x_n, \end{aligned}$$

and $\mathcal{A}_i | \mathcal{A}_{i+1}$ $(1 \le i < n)$; the $\mathcal{A}_i$ are unique and called the invariant factors of Ω in Λ. We use $\{\Lambda : \Omega\}$ to refer to these invariant factors, and we write $\{\Lambda : \Omega\} = (\mathcal{A}_1, \dots, \mathcal{A}_n)$.

When analyzing the action of Hecke operators on Fourier coefficients, we sum over lattices Λ where $\mathcal{P}\Lambda \subseteq \Omega \subseteq \mathcal{P}^{-1}\Lambda$, $\mathcal{P}$ a prime ideal and Λ a fixed reference lattice of rank n. By the Invariant Factor Theorem, we have sublattices Λ_i so that

$$\begin{aligned} \Lambda &= \Lambda_0 \oplus \Lambda_1 \oplus \Lambda_2, \\ \Omega &= \mathcal{P}\Lambda_0 \oplus \Lambda_1 \oplus \mathcal{P}^{-1}\Lambda_2. \end{aligned}$$

So for instance, $r_0 = \operatorname{rank} \Lambda_0$ is the multiplicity of $\mathcal{P}$ among the invariant factors $\{\Lambda : \Omega\}$, denoted $r_0 = \operatorname{mult}_{\{\Lambda:\Omega\}}(\mathcal{P})$.

We will also need to consider $(\Lambda^{\mathcal{J}} \cap \Omega^{\mathcal{J}})/\mathcal{P}(\Lambda^{\mathcal{J}} + \Omega^{\mathcal{J}}) \approx \Lambda_1^{\mathcal{J}}/\mathcal{P}\Lambda_1^{\mathcal{J}}$. We will only be considering even integral $\Lambda^{\mathcal{J}}$. Thus Q induces a quadratic form $\frac{1}{2}\alpha Q$ on $\Lambda_1^{\mathcal{J}}/\mathcal{P}\Lambda_1^{\mathcal{J}}$ defined by

$$\frac{1}{2}\alpha Q(x + \mathcal{P}\Lambda_1) = \frac{1}{2}\alpha Q(x) + \mathcal{P} \in \mathcal{O}/\mathcal{P}$$

where $\alpha \in \mathbb{K}$ has been fixed so that $\alpha\mathcal{O}_{\mathcal{P}} = \mathcal{J}\mathcal{O}_{\mathcal{P}}$. Since $Q(\Lambda_1) \subseteq 2\mathcal{J}^{-1}$, this gives us a quadratic form on the $\mathcal{O}/\mathcal{P}$-space $\Lambda_1^{\mathcal{J}}/\mathcal{P}\Lambda_1^{\mathcal{J}}$. Note that the structure of the quadratic space $\Lambda_1^{\mathcal{J}}/\mathcal{P}\Lambda_1^{\mathcal{J}}$ is independent of the choice of α.

Definition. Let $f \in \mathcal{M}_k$. Given any even integral positive semi-definite lattice $\Lambda^{\mathcal{J}}$ with $\Lambda = \mathcal{I}_1 x_1 \oplus \cdots \oplus \mathcal{I}_n x_n$, we set

$$c(\Lambda^{\mathcal{J}}) = c_F(\Lambda^{\mathcal{J}}) = c_f(T) \cdot N(\mathcal{I}_1 \cdots \mathcal{I}_n)^k N(\mathcal{J})^{nk/2}$$

where $f \in \mathcal{M}_k(\Gamma(\mathcal{I}_1, \dots, \mathcal{I}_n; \mathcal{J}))$ is a representative of the component of F corresponding to $\operatorname{cls}(\mathcal{I}_1 \cdots \mathcal{I}_n)$, $\operatorname{clx}^+\mathcal{J}$, and $T = \left(B(x_i, x_j) \quad \right)$. If k is odd, we assume Λ is also equipped with an orientation.

Proposition 2.2. *The "Fourier coefficient" $c(\Lambda^{\mathcal{J}})$ is well-defined.*

Proof. First, suppose we also have $\Lambda' = \mathcal{I}_1 y_1 \oplus \cdots \oplus \mathcal{I}_n y_n$, and $\Lambda = \Lambda'$. Take $M = (\alpha_{ij})$ so that

$$(y_1 \ldots y_n) = (x_1 \ldots x_n)M.$$

Hence $(B(y_i, y_j)) = {}^tMTM$ (recall $T = (B(x_i, x_j))$). Note that

$$\mathcal{I}_j y_j = \sum_i \alpha_{ij} \mathcal{I}_j x_i \subseteq \Lambda,$$

so $\alpha_{ij} \in \mathcal{I}_i \mathcal{I}_j^{-1}$. Also, since $\operatorname{vol}\Lambda = \operatorname{vol}\Lambda'$, it follows that $\det M \in \mathcal{O}^\times$. (Recall that if k is odd then Λ has an orientation, and $\det M$ must also be totally positive). Thus $\left(\begin{smallmatrix} M & \\ & {}^tM^{-1} \end{smallmatrix}\right) \in \Gamma(\mathcal{I}_1, \ldots, \mathcal{I}_n; \mathcal{J})$, and so

$$f = f \left| \begin{pmatrix} M & \\ & {}^tM^{-1} \end{pmatrix} \right. .$$

Hence

$$\begin{aligned} f(\tau) &= \sum_T c_f(T) e\{T\tau\} \\ &= N(\det M)^k \sum_T c_f(T) e\{TM\tau\, {}^tM\} \\ &= \sum_T c_f(T) e\{{}^tMTM\tau\} \end{aligned}$$

(recall that $\det M$ is a unit, and that if k is odd, a totally positive unit). Thus $c_f({}^tMTM) = N(\det M)^k\ c_f(T) = c_f(T)$, and so

$$\begin{aligned} c(\Lambda^{\mathcal{J}}) &= c_f(T) N(\mathcal{I}_1 \cdots \mathcal{I}_n)^k N(\mathcal{J})^{nk/2} \\ &= c_f({}^tMTM) N(\mathcal{I}_1 \cdots \mathcal{I}_n)^k N(\mathcal{J})^{nk/2} = c(\Lambda'^{\mathcal{J}}). \end{aligned}$$

Thus the definition of $c(\Lambda^{\mathcal{J}})$ is independent of the choice of basis relative to the coefficient ideals $\mathcal{I}_1, \ldots, \mathcal{I}_n$ and the scaling ideal $\mathcal{J}$ (so here $\mathcal{I}_1, \ldots, \mathcal{I}_n$ and $\mathcal{J}$ are fixed).

Next, fix $\mathcal{J}$ and suppose $\Lambda = \mathcal{I}_1 x_1 \oplus \cdots \oplus \mathcal{I}_n x_n = \mathcal{I}'_1 y'_1 \oplus \cdots \oplus \mathcal{I}'_n y'_n$. Then by [6, 81:8], $\mathcal{I}'_1 \cdots \mathcal{I}'_n \in \operatorname{cls}(\mathcal{I}_1 \cdots \mathcal{I}_n)$. Thus, as we have seen, $M_k(\Gamma(\mathcal{I}_1, \ldots, \mathcal{I}_n; \mathcal{J})) \simeq M_k(\Gamma(\mathcal{I}'_1, \ldots, \mathcal{I}'_n; \mathcal{J}))$ via an appropriate composition of the maps U_i, V_{ij}; the action of this composition is given by

$$f \mapsto f' = f \left| \begin{pmatrix} M & \\ & {}^tM^{-1} \end{pmatrix} \right.$$

where $M \in \left(\mathcal{I}_i \mathcal{I}_j^{-1}\right)$ and $(\det M)\mathcal{I}'_1 \cdots \mathcal{I}'_n = \mathcal{I}_1 \cdots \mathcal{I}_n$. Also,

$$f'(\tau) = N(\det M)^k f(M\tau\, {}^tM) = N(\det M)^k \sum_T c_f(T) \mathrm{e}\{{}^tMTM\tau\},$$

so $c_{f'}({}^tMTM) = N(\det M)^k c_f(T)$. Set $(y_1 \cdots y_n) = (x_1 \cdots x_n)M$; thus $\mathcal{O}y_1 \oplus \cdots \oplus \mathcal{O}y_n \simeq {}^tMTM$. We claim that with $\Lambda' = \mathcal{I}'_1 y_1 \oplus \cdots \oplus \mathcal{I}'_n y_n$, $\Lambda' = \Lambda$. We know that writing M as (α_{ij}),

$$y_j = \sum_{i=1}^{n} \alpha_{ij} x_i \in (\mathcal{I}'_j)^{-1}(\mathcal{I}_1 x_1 \oplus \cdots \oplus \mathcal{I}_n x_n)$$

and so $\Lambda' \subseteq \Lambda$. Since $\mathrm{norm}(\Lambda) = (\mathcal{I}_1 \cdots \mathcal{I}_n)^2 \cdot \det T = (\mathcal{I}'_1 \cdots \mathcal{I}'_n)^2 \cdot \det({}^tMTM) = \mathrm{norm}(\Lambda')$, we have $\Lambda' = \Lambda$. So

$$\begin{aligned} c(\Lambda'^{\mathcal{J}}) &= c_{f'}({}^tMTM)\ N(\mathcal{I}'_1 \cdots \mathcal{I}'_n)^k\ N(\mathcal{J})^{nk/2} \\ &= c_f(T)\ N(\det M)^k\ N(\mathcal{I}'_1 \cdots \mathcal{I}'_n)^k\ N(\mathcal{J})^{nk/2} \\ &= c_f(T)\ N(\mathcal{I}_1 \cdots \mathcal{I}_n)^k\ N(\mathcal{J})^{nk/2} \\ &= c(\Lambda^{\mathcal{J}}). \end{aligned}$$

Thus by our assumption, $\mathcal{I}'_1 y_1 \oplus \cdots \oplus \mathcal{I}'_n y_n = \mathcal{I}'_1 y'_1 \oplus \cdots \oplus \mathcal{I}'_n y'_n$, which reduces the problem to the preceding case. Hence our definition of $c(\Lambda^{\mathcal{J}})$ is independent of the choice of classes $\mathcal{I}_1, \ldots, \mathcal{I}_n$ so that $\mathrm{cls}(\mathcal{I}_1 \cdots \mathcal{I}_n)$ is as prescribed (so here $\mathrm{cls}(\mathcal{I}_1 \cdots \mathcal{I}_n), \mathcal{J}$ are fixed).

Finally, suppose $\mathcal{J}' \in \mathrm{clx}^+ \mathcal{J}$. Thus $\mathcal{J}' = \alpha \mathcal{I}^2 \mathcal{J}$ for some fractional ideal $\mathcal{I}$ and some $\alpha \gg 0$. Say $\Lambda = \mathcal{I}_1 x_1 \oplus \cdots \oplus \mathcal{I}_n x_n$. We have agreed previously to identify $\Lambda^{\mathcal{I}^2}$ and $\mathcal{I}\Lambda$, so that $c_f(\Lambda^{\alpha\mathcal{I}^2\mathcal{J}}) = c_f(\mathcal{I}\Lambda^{\alpha\mathcal{J}})$, whether we think of f as associated to $\Gamma(\mathcal{I}_1, \ldots, \mathcal{I}_n; \alpha\mathcal{I}^2\mathcal{J})$ or to $\Gamma(\mathcal{I}\mathcal{I}_1, \ldots, \mathcal{I}\mathcal{I}_n; \alpha\mathcal{J})$ (remember, these are two names for the same group). So suppose $\mathcal{J}' = \alpha\mathcal{J}$, $\alpha \gg 0$. We know $\mathcal{M}_k(\Gamma(\mathcal{I}_1, \ldots, \mathcal{I}_n; \mathcal{J})) \simeq \mathcal{M}_k(\Gamma(\mathcal{I}_1, \ldots, \mathcal{I}_n; \alpha\mathcal{J}))$ via

$$f \mapsto f' = f \left| \begin{pmatrix} \alpha^{-1} I & \\ & I \end{pmatrix} \right. .$$

Thus

$$f'(\tau) = N(\alpha)^{-nk/2} f(\alpha^{-1}\tau) = N(\alpha)^{-nk/2} \sum_T c_f(T) e\{\alpha^{-1} T\tau\}.$$

Consequently $c_{f'}(\alpha^{-1}T) = N(\alpha)^{-nk/2}\ c_f(T)$. Set $\Lambda' = \mathcal{I}_1 x_1 \oplus \cdots \oplus \mathcal{I}_n x_n$ equipped with the quadratic form $\alpha^{-1}T$ (so $(\Lambda')^{\alpha\mathcal{J}}$ is an integral lattice). Then we have

$$\begin{aligned} c(\Lambda^{\mathcal{J}}) &= c_f(T)\ N(\mathcal{I}_1 \cdots \mathcal{I}_n)^k\ N(\mathcal{J})^{nk/2} \\ &= c_{f'}(\alpha^{-1}T)\ N(\mathcal{I}_1 \cdots \mathcal{I}_n)^k\ N(\alpha\mathcal{J})^{nk/2} \\ &= c((\Lambda')^{\alpha\mathcal{J}}). \end{aligned}$$

Thus the definition of $c(\Lambda^{\mathcal{J}})$ is also independent of the choice of the representative for $\mathrm{clx}^+ \mathcal{J}$. □

3. Eigenspaces

The $V_i(\mathcal{Q})$ operators are lifts of the Hilbert modular form operators that Eichler called $V(\mathcal{Q}^{-1})$ [3] and Shimura called $S(\mathcal{Q})$ [7]. We now introduce another lift of these operators, and following Shimura [7] (where $n = 1$), we decompose $\mathcal{M}_k$ into eigenspaces for these new operators.

Definition. Let $\mathcal{Q}$ be a fractional ideal, and choose

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{pmatrix} \mathcal{Q} & \mathcal{Q}^{-1}\mathcal{I}_\ell^2\mathcal{J}\partial^{-1} \\ \mathcal{Q}\mathcal{I}_\ell^{-2}\mathcal{J}^{-1}\partial & \mathcal{Q}^{-1} \end{pmatrix}$$

with $ad - bc = 1$. Set

$$M = \begin{pmatrix} I_{\ell-1} & & & 0_{\ell-1} & & \\ & a & & & b & \\ & & I_{n-\ell} & & & 0 \\ 0_{\ell-1} & & & I_{\ell-1} & & \\ & c & & & d & \\ & & 0 & & & I_{n-\ell} \end{pmatrix}.$$

Then $M\Gamma(\mathcal{I}_1,\dots,\mathcal{I}_n;\mathcal{J})M^{-1} = \Gamma(\mathcal{I}'_1,\dots,\mathcal{I}'_n;\mathcal{J})$ where

$$\mathcal{I}'_j = \begin{cases} \mathcal{I}_j & \text{if } j \neq \ell, \\ \mathcal{Q}^{-1}\mathcal{I}_\ell & \text{if } j = \ell. \end{cases}$$

Thus $S_\ell(\mathcal{Q}) : \mathcal{M}_k(\Gamma(\mathcal{I}_1,\dots,\mathcal{I}_n;\mathcal{J})) \to \mathcal{M}_k(\Gamma(\mathcal{I}'_1,\dots,\mathcal{I}'_n;\mathcal{J}))$ is an isomorphism where we define

$$f|S_\ell(\mathcal{Q}) = f|M.$$

Proposition 3.1. *With $\mathcal{Q},\mathcal{P}$ fractional ideals and $\alpha \in \mathbb{K}^\times$, $S_\ell(\mathcal{Q})$ commutes with $U_i(\alpha)$, $W(\alpha)$, and $V_{ij}(\mathcal{P})$. Further, $S_i(\mathcal{Q})V_{ij}(\mathcal{Q}) = S_j(\mathcal{Q})$ and $U_i(\alpha^{-1})S_i(\mathcal{Q}) = S_i(\alpha\mathcal{Q})$.*

Proof. Keeping in mind the various domains for different incarnations of our functions, it is easy to show $S_\ell(\mathcal{Q})$ commutes with $U_i(\alpha), W(\alpha)$.

To show $S_\ell(\mathcal{Q}), V_{ij}(\mathcal{P})$ commute, it suffices to show $S_\ell(\mathcal{Q})$, $V_i(\mathcal{P})$ commute (recall how V_{ij} is defined). When $\ell \neq i, i+1$, it is easy to see the matrices giving the actions of $S_\ell(\mathcal{Q}), V_i(\mathcal{P})$ commute. As we explain below, we can reduce our attention to the case $n = 2$.

Suppose $\ell = i$ or $i+1$; for the sake of clarity, let us first look at the case where $i = 1, \ell = 1$ or 2. Then the action of each operator is given by a matrix of the form

$$\begin{pmatrix} A & & B & \\ & I & & 0 \\ C & & D & \\ & 0 & & I \end{pmatrix}$$

where A, B, C, D are 2×2 matrices. Since the product of such matrices (and their inverses) is again of this form, it suffices to restrict our attention to the submatrices $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$.

First consider $n = 2$, $i = \ell = 1$. Choose

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{pmatrix} \mathcal{Q} & \mathcal{Q}\mathcal{P}^2\mathcal{I}_1^2\mathcal{J}\partial^{-1} \\ (\mathcal{Q}\mathcal{I}_1^2\mathcal{J})^{-1}\partial & \mathcal{Q}^{-1} \end{pmatrix}$$

so that $ad - bc = 1$. Choose

$$A \in \begin{pmatrix} \mathcal{P}^{-1} & \mathcal{P}\mathcal{Q}\mathcal{I}_1\mathcal{I}_2^{-1} \\ \mathcal{P}^{-1}\mathcal{I}_1^{-1}\mathcal{I}_2 & \mathcal{P} \end{pmatrix}$$

so that $\det A = 1$. (Note that these choices are possible, even when $\mathcal{Q} = \mathcal{P}$.) Then

$$M = \begin{pmatrix} a & & b & \\ & 1 & & 0 \\ c & & d & \\ & 0 & & 1 \end{pmatrix}$$

gives the action of both

$$S_1(\mathcal{Q}) : \mathcal{M}_k(\Gamma(\mathcal{Q}\mathcal{I}_1, \mathcal{I}_2; \mathcal{J})) \to \mathcal{M}_k(\Gamma(\mathcal{I}_1, \mathcal{I}_2; \mathcal{J}))$$

and

$$S_1(\mathcal{Q}) : \mathcal{M}_k(\Gamma(\mathcal{P}\mathcal{Q}\mathcal{I}_1, \mathcal{P}^{-1}\mathcal{I}_2; \mathcal{J})) \to \mathcal{M}_k(\Gamma(\mathcal{P}\mathcal{I}_1, \mathcal{P}^{-1}\mathcal{I}_2; \mathcal{J})).$$

Similarly, $N = \begin{pmatrix} A & \\ & {}^tA^{-1} \end{pmatrix}$ gives the action of both

$$V_1(\mathcal{Q}) : \mathcal{M}_k(\Gamma(\mathcal{Q}\mathcal{I}_1, \mathcal{I}_2; \mathcal{J})) \to \mathcal{M}_k(\Gamma(\mathcal{P}\mathcal{Q}\mathcal{I}_1, \mathcal{P}^{-1}\mathcal{I}_2; \mathcal{J}))$$

and

$$V_1(\mathcal{Q}) : \mathcal{M}_k(\Gamma(\mathcal{I}_1, \mathcal{I}_2; \mathcal{J})) \to \mathcal{M}_k(\Gamma(\mathcal{P}\mathcal{I}_1, \mathcal{P}^{-1}\mathcal{I}_2; \mathcal{J})).$$

A simple (but tiresome) check shows $MNM^{-1}N^{-1} \in \Gamma(\mathcal{Q}\mathcal{I}_1, \mathcal{I}_2; \mathcal{J})$, which implies that $S_1(\mathcal{Q}), V_1(\mathcal{P})$ commute when $n = 2$. For general n, we have $S_1(\mathcal{Q})V_1(\mathcal{P})S_1(\mathcal{Q}^{-1})V_1(\mathcal{P}^{-1})$ represented by a matrix of the form

$$M' = \begin{pmatrix} A & & B & \\ & I & & 0 \\ C & & D & \\ & 0 & & I \end{pmatrix}$$

where $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma(\mathcal{Q}\mathcal{I}_1, \mathcal{I}_2; \mathcal{J})$. Thus $M' \in \Gamma(\mathcal{Q}\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n; \mathcal{J})$, and $S_1(\mathcal{Q}), V_1(\mathcal{P})$ commute for general n. Similarly, $S_2(\mathcal{Q}), V_1(\mathcal{P})$ commute for $n = 2$, and thus for general n.

For general n, ℓ, i with $\ell = i$ or $i + 1$, the matrices giving the action of $S_\ell(\mathcal{Q}), V_i(\mathcal{P})$ are of the form

$$\begin{pmatrix} I_{i-1} & & & 0_{i-1} & & \\ & A & & & B & \\ & & I & & & 0 \\ 0_{i-1} & & & I_{i-1} & & \\ & C & & & D & \\ & & 0 & & & I \end{pmatrix},$$

where A, B, C, D are 2×2 matrices. So again the problem reduces to showing $S_\ell(\mathcal{Q}), V_i(\mathcal{P})$ ($\ell = i$ or $i + 1$) commute when $n = 2$ (which we have done).

To see that $S_i(\mathcal{Q})V_{ij}(\mathcal{Q}) = S_j(\mathcal{Q})$, it again suffices to consider $j = i + 1$. First, fix a group $\Gamma = \Gamma(\mathcal{I}_1, \ldots, \mathcal{I}_n; \mathcal{J})$. From our definitions, we have a product of three matrices giving the action of $S_i(\mathcal{Q})V_i(\mathcal{Q})S_{i+1}(\mathcal{Q}^{-1})$ on $\mathcal{M}_k(\Gamma)$. One easily verifies that the conditions on these matrices ensure the product for $S_i(\mathcal{Q})V_i(\mathcal{Q})S_{i+1}(\mathcal{Q}^{-1})$ lies in Γ. Finally, one verifies $U_i(\alpha^{-1})S_i(\mathcal{Q}) = S_i(\alpha\mathcal{Q})$ by matrix multiplication. $\square$

This shows that the $S_\ell(\mathcal{Q})$ act on $\mathcal{M}_k$, where, for $F \in \mathcal{M}_k$, $F \sim (\ldots, f_i, \ldots)$, $F|S_\ell(\mathcal{Q}) \sim (\ldots, f_i|S_\ell(\mathcal{Q}), \ldots)$. It also shows that $S_i(\mathcal{Q}), S_j(\mathcal{Q})$ are equivalent on $\mathcal{M}_k$, and so we simply refer to this operator on $\mathcal{M}_k$ as $S(\mathcal{Q})$. Furthermore, on $\mathcal{M}_k$, $S(\mathcal{Q}) = S(\alpha\mathcal{Q})$ for all $\alpha \in \mathbb{K}^\times$; since we also know $S(*)$ is multiplicative, the map $\mathrm{cls}\,\mathcal{I} \mapsto S(\mathcal{I})$ gives a group action of the ideal class group on $\mathcal{M}_k$.

Proposition 3.2. $\mathcal{M}_k = \oplus_\chi \mathcal{M}_k(\chi)$ *where* χ *varies over all characters of the ideal class group, and*

$$\mathcal{M}_k(\chi) = \{F \in \mathcal{M}_k \, : \, F|S(\mathcal{Q}) = \chi(\mathcal{Q})F \text{ for all } \mathcal{Q}\}.$$

Proof. First notice that $\mathcal{M}_k(\chi) \cap \mathcal{M}_k(\psi) = \{0\}$ if $\chi \neq \psi$. To prove this take $F \in \mathcal{M}_k(\chi) \cap \mathcal{M}_k(\psi)$. Then $\chi(\mathcal{Q})F = F|S(\mathcal{Q}) = \psi(\mathcal{Q})F$ for all $\mathcal{Q}$. Since $\chi \neq \psi$ there is a $\mathcal{Q}$ such that $\chi(\mathcal{Q}) \neq \psi(\mathcal{Q})$. Therefore $F = 0$.

For ψ an ideal class character, let $G_\psi = \frac{1}{h}\sum_{\mathrm{cls}\,\mathcal{I}} \overline{\psi}(\mathcal{I})\; F|S(\mathcal{I})$ where h is the class number of $\mathbb{K}$. Note that

$$\sum_\psi G_\psi = \frac{1}{h}\sum_{\mathrm{cls}\,\mathcal{I}}\left(\sum_\psi \overline{\psi}(\mathcal{I})\right) F|S(\mathcal{I}) = F$$

since

$$\sum_\psi \overline{\psi}(\mathcal{I}) = \begin{cases} h & \text{if } \mathrm{cls}\,\mathcal{I} = \mathrm{cls}\,\mathcal{O}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $F = \sum_\psi G_\psi$.

Next notice that

$$\begin{aligned} G_\psi|S(\mathcal{Q}) &= \frac{1}{h}\sum_{\mathrm{cls}\,\mathcal{I}} \overline{\psi}(\mathcal{I})F|S(\mathcal{I})|S(\mathcal{Q}) \\ &= \frac{1}{h}\sum_{\mathrm{cls}\,\mathcal{I}} \overline{\psi}(\mathcal{I})F|S(\mathcal{I}\mathcal{Q}) \\ &= \frac{1}{h}\psi(\mathcal{Q})\sum_{\mathrm{cls}\,\mathcal{I}} \overline{\psi}(\mathcal{I}\mathcal{Q})F|S(\mathcal{I}\mathcal{Q}) \\ &= \psi(\mathcal{Q})G_\psi. \end{aligned}$$

Thus $G_\psi \in \mathcal{M}_k(\psi)$, and hence $F \in \oplus_\chi \mathcal{M}_k(\chi)$. $\square$

4. Hecke Operators

We begin by defining the Hecke operators. Then we show that they act on each $\mathcal{M}_k(\chi)$. After this, we describe how to find a set of coset representatives giving the action of the operators. Finally, in the next section we analyze the action of the operators on Fourier coefficients attached to even integral lattices, proving our main theorem.

Definition. Let $\mathcal{P}$ a prime ideal; set $\Gamma = \Gamma(\mathcal{I}_1, \dots, \mathcal{I}_n; \mathcal{J})$ and $\Gamma' = \Gamma(\mathcal{I}_1, \dots, \mathcal{I}_n; \mathcal{P}\mathcal{J})$. We define the Hecke operator $T(\mathcal{P}) : \mathcal{M}_k(\Gamma') \to \mathcal{M}_k(\Gamma)$ by

$$F|T(\mathcal{P}) = N(\mathcal{P})^{n(k-n-1)/2} \sum_{\gamma} F|\gamma$$

where γ runs over a complete set of coset representatives for $(\Gamma' \cap \Gamma)\backslash\Gamma$. Note that Γ' is the formal conjugate of Γ by the matrix $\delta = \left(\begin{smallmatrix} \mathcal{P}I_n & \\ & I_n \end{smallmatrix}\right)$. When $\mathbb{K} = \mathbb{Q}$, we define $T(p)$ on $\mathcal{M}_k(\Gamma)$ by

$$f|T(p) = p^{n(k-n-1)/2} \sum_{\gamma} f|\delta^{-1}\gamma$$

where γ runs over a complete set of coset representatives for $(\Gamma' \cap \Gamma)\backslash\Gamma$, $\Gamma' = \delta\Gamma\delta^{-1}$, $\delta = \left(\begin{smallmatrix} pI & \\ & I \end{smallmatrix}\right)$. (This normalization of $T(p)$ is standard, and as with the standard normalization of the degree 1 Hecke operator $T(p)$, the purpose of the normalization is to force the coefficient to be 1 on the "lead" term in the expression for the Λth coefficient of $F|T(p)$.)

Now fix $1 \le j \le n$; let $\Gamma'_j = \Gamma(\mathcal{P}\mathcal{I}_1, \dots, \mathcal{P}\mathcal{I}_j, \mathcal{I}_{j+1}, \dots, \mathcal{I}_n; \mathcal{J})$. We define the Hecke operators $T_j(\mathcal{P}^2) : \mathcal{M}_k(\Gamma'_j) \to \mathcal{M}_k(\Gamma)$ by

$$F|T_j(\mathcal{P}^2) = \sum_{\gamma} F|\gamma$$

where γ runs over a complete set of coset representatives for $(\Gamma'_j \cap \Gamma)\backslash\Gamma$.

Note that Γ'_j is the formal conjugate of Γ by $\mathrm{diag}(\mathcal{P}I_j, I_{n-j}, \mathcal{P}^{-1}I_j, I_{n-j})$. When $\mathbb{K} = \mathbb{Q}$, $\mathcal{P} = p\mathbb{Z}$, we define $T_j(\mathcal{P}^2) = T_j(p^2)$ on $\mathcal{M}_k(\Gamma)$ by

$$f|T_j(\mathcal{P}^2) = \sum_{\gamma} f|\delta^{-1}\gamma$$

where $\delta = \mathrm{diag}(pI_j, I_{n-j}, \frac{1}{p}I_j, I_{n-j})$, $\Gamma' = \delta\Gamma\delta^{-1}$, and γ runs over a complete set of coset representatives for $(\Gamma' \cap \Gamma)\backslash\Gamma$. (We introduce a normalization later.)

Proposition 4.1. *The operators $T(\mathcal{P}), T_j(\mathcal{P}^2)$ commute with $U_i(\alpha), W(\beta), V_{i\ell}(\mathcal{Q})$, and $S_i(\mathcal{Q})$ where $\alpha, \beta \in \mathbb{K}^\times$ with $\beta \gg 0$, and $\mathcal{Q}$ is a fractional ideal. Thus $T(\mathcal{P})$, $T_j(\mathcal{P}^2)$ act on $\mathcal{M}_k(\chi)$ (as defined in Proposition* 3.2*).*

Proof. To show $T_j(\mathcal{P}^2)$ commutes with the $V_{i\ell}(\mathcal{Q})$, it suffices to show it commutes with $V_i(\mathcal{Q})$. Take

$$A \in \begin{pmatrix} \mathcal{Q}^{-1} & \mathcal{I}_i\mathcal{I}_j^{-1}\mathcal{Q}\mathcal{P} \\ \mathcal{I}_i^{-1}\mathcal{I}_j\mathcal{Q}^{-1}\mathcal{P} & \mathcal{Q} \end{pmatrix}$$

with $\det A = 1$, and set

$$\mathcal{I}_i' = \begin{cases} \mathcal{P}\mathcal{I}_i & \text{if } i \le j, \\ \mathcal{I}_i & \text{otherwise.} \end{cases}$$

Then with

$$M = \begin{pmatrix} I_{i-1} &&&&& \\ & A &&&& \\ && I_{n-i-1} &&& \\ &&& I_{i-1} && \\ &&&& {}^tA^{-1} & \\ &&&&& I_{n-i-1} \end{pmatrix},$$

M gives the action of $V_i(\mathcal{Q}) : \mathcal{M}_k(\Gamma) \to \mathcal{M}_k(M^{-1}\Gamma M)$ and of $V_i(\mathcal{Q}) : \mathcal{M}_k(\Gamma') \to \mathcal{M}_k(M^{-1}\Gamma'\mathcal{M})$ (note that these are the appropriate codomains). Now let $\{\gamma\}$ be a complete set of coset representatives for $(\Gamma' \cap \Gamma)\backslash\Gamma$. Thus $\{M^{-1}\gamma M\}$ is a complete set of coset representatives for $(M^{-1}\Gamma' M \cap M^{-1}\Gamma M)\backslash M^{-1}\Gamma M$. Hence for $f \in \mathcal{M}_k(\Gamma')$,

$$\begin{aligned} f|T_j(\mathcal{P}^2)V_i(\mathcal{Q}) &= \sum_\gamma f|\gamma|M \\ &= \sum_\gamma f|M|M^{-1}\gamma M \\ &= f|V_i(\mathcal{Q})T_j(\mathcal{P}^2). \end{aligned}$$

Similarly, to show $T_j(\mathcal{P}^2)$ commutes with $S_i(\mathcal{Q})$, choose

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{pmatrix} \mathcal{Q} & \mathcal{Q}^{-1}\mathcal{I}_i^2\mathcal{P}^2\mathcal{J}\partial^{-1} \\ \mathcal{Q}\mathcal{I}_i^{-2}\mathcal{J}^{-1}\partial & \mathcal{Q}^{-1} \end{pmatrix}$$

so that $ad - bc = 1$. Then $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ lifts to a matrix M so that M gives the action of $S_i(\mathcal{Q}) : \mathcal{M}_k(\Gamma) \to \mathcal{M}_k(M^{-1}\Gamma M)$ and of $S_i(\mathcal{Q}) : \mathcal{M}_k(\Gamma') \to \mathcal{M}_k(M^{-1}\Gamma' M)$. Thus

$$\begin{aligned} f|T_j(\mathcal{P}^2)S_i(\mathcal{Q}) &= \sum_\gamma f|\gamma|M \\ &= \sum_\gamma f|M^{-1}|M^{-1}\gamma M \\ &= f|S_i(\mathcal{Q})T_j(\mathcal{P}^2). \end{aligned}$$

Similar but simpler arguments show that $T_j(\mathcal{P}^2)$ commutes with $U_i(\alpha), W(\beta)$, and that $T(\mathcal{P})$ commutes with $V_i(\mathcal{Q}), S_i(\mathcal{Q}), U_i(\alpha), W(\beta)$. $\square$

Proposition 4.2. *For $f' \in \mathcal{M}_k(\Gamma')$, we have*

$$f'|T_j(\mathcal{P}^2) = \sum_{\Omega,\Lambda_1,Y} f'|S^{(j)}(\Omega) \begin{pmatrix} I & Y \\ & I \end{pmatrix} \begin{pmatrix} C^{-1} & \\ & {}^tC \end{pmatrix}.$$

Here Ω varies over all lattices such that $\mathcal{P}\Lambda \subseteq \Omega \subseteq \mathcal{P}^{-1}\Lambda$, $\overline{\Lambda}_1$ varies over all codimension $n-j$ subspaces of $\Omega \cap \Lambda/\mathcal{P}(\Omega + \Lambda)$, $C = C(\Omega, \Lambda_1)$. With

$r_0 = \mathrm{mult}_{\{\Lambda:\Omega\}}(\mathcal{P}),\ m_1 = \mathrm{mult}_{\{\Lambda:\Omega\}}(\mathcal{O}),\ r_1 = m_1 - n + j,\ \mu \in \mathcal{P}^{-1} - \mathcal{O}$ *fixed,*

$$\Omega = \Lambda C \mathrm{diag}(\mathcal{P} I_{r_0},\ I_{r_1},\ \mathcal{P}^{-1} I_{r_2},\ I_{n-j}),$$

$$S^{(j)}(\Omega) = \left(\prod_{i=r_0+1}^{r_0+r_1} S_i(\mathcal{P}) \right) \left(\prod_{i=r_0+r_1+1}^{j} S_i(\mathcal{P}^2) \right),$$

$$Y = \begin{pmatrix} W_0 & W_2 & 0 & W_3 \\ {}^tW_2 & \mu W_1 & & \\ 0 & & & \\ {}^tW_3 & & & \end{pmatrix}$$

with $Y \in (\mathcal{I}_i \mathcal{I}_j \mathcal{J} \partial^{-1})$, W_0 *varying modulo* $\mathcal{P}^2$, W_1, W_2, W_3 *varying modulo* $\mathcal{P}$ *with* $\mathcal{P}$ *not dividing* $\det W_1$. (*Recall that* ∂ *is the different of* $\mathbb{K}$.) *Here* W_0 *is* $r_0 \times r_0$ *and symmetric,* W_1 *is* $r_1 \times r_1$ *and symmetric,* W_2 *is* $r_0 \times r_1$, W_3 *is* $r_0 \times (n-j)$.

Proof. Let $\mathcal{P}$ be a prime ideal, and fix j, $1 \le j \le n$. We essentially follow the algorithm presented in [5] to find a set of coset representatives giving the action of

$$T_j(\mathcal{P}^2) : \mathcal{M}_k(\Gamma(\mathcal{P}\mathcal{I}_1, \ldots, \mathcal{P}\mathcal{I}_j, \mathcal{I}_{j+1}, \ldots, \mathcal{I}_n; \mathcal{J})) \to \mathcal{M}_k(\Gamma(\mathcal{I}_1, \ldots, \mathcal{I}_n; \mathcal{J})).$$

For convenience, we will take $\mathcal{I}_i = \mathcal{O}$ for $1 \le i \le j$, $\mathcal{I}_i = \mathcal{O}$ for $j < i < n$; also, we take $\mathcal{I} = \mathcal{I}_n$ and $\mathcal{J}\partial^{-1}$ to be integral ideals relatively prime to $\mathcal{P}$ (recall that the equivalence class of $\mathcal{M}_k(\Gamma)$ is determined by cls $\mathcal{I}$, $\mathrm{cls}^+ \mathcal{J}$, and ∂ is the different of $\mathbb{K}$). Note that this allows us to choose μ relatively prime to $\mathcal{I}_1 \cdots \mathcal{I}_n \mathcal{J}\partial^{-1} = \mathcal{I}\mathcal{J}\partial^{-1}$.

Choose $M \in \Gamma(\mathcal{O}, \ldots, \mathcal{O}, \mathcal{I}; \mathcal{J})$, and let $M_j = (A|B)$ denote the top j rows of M with A, B $j \times n$ matrices. Let $\Lambda = \mathcal{O}_{\mathcal{P}} x_1 \oplus \cdots \oplus \mathcal{O}_{\mathcal{P}} x_n$ be a reference lattice.

Step 1. Let

$$\Omega_0 = \ker(\Lambda \to \Lambda(A) \bmod \mathcal{P}\mathcal{O}_{\mathcal{P}})$$

where $A = (a_1 \cdots a_n)$ and $\Lambda \to \Lambda(A) \bmod \mathcal{P}\mathcal{O}_{\mathcal{P}}$ denotes the map that takes x_i to $\overline{a}_i$ (which is a $1 \times j$ matrix with entries in $\mathcal{O}_{\mathcal{P}}/\mathcal{P}\mathcal{O}_{\mathcal{P}}$). Note that the $r_0 = \mathrm{rank}_{\mathcal{P}} \Lambda(A)$ is at most j since A is a $j \times n$ matrix.

We claim there is a matrix $\begin{pmatrix} C_0 & \\ & {}^tC_0^{-1} \end{pmatrix} \in \Gamma(\mathcal{O}, \ldots, \mathcal{O}, \mathcal{I}; \mathcal{J})$ such that $\Omega_0 = \Lambda C_0 \begin{pmatrix} \mathcal{P} I_{r_0} & \\ & I_{n-r_0} \end{pmatrix}$ and $(A|B) \begin{pmatrix} C_0 & \\ & {}^tC_0^{-1} \end{pmatrix} = (A'|B')$ with $a'_1, \ldots, a'_n \equiv 0 \pmod{\mathcal{P}}$.

First, write $A = (a_1 \cdots a_n)$ and consider the rank modulo $\mathcal{P}\mathcal{O}_{\mathcal{P}}$ of $(a_1 \cdots a_n)$. Let E_1 be an $(n-1) \times (n-1)$ invertible matrix (i.e. a change of basis matrix) so that

$$(a_1 \cdots a_n) E_1 = (a'_1 \cdots a'_n)$$

with $a'_1, \ldots, a'_{r_0}$ linearly independent modulo $\mathcal{P}\mathcal{O}_{\mathcal{P}}$ and $a'_{r_0+1} \equiv \cdots \equiv a'_{n-1} \equiv 0 \pmod{\mathcal{P}\mathcal{O}_{\mathcal{P}}}$. Note that $G_1 = \begin{pmatrix} E_1 & & \\ & {}^tE_1^{-1} & \\ & & 1 \end{pmatrix} \in \Gamma(\mathcal{O}, \ldots, \mathcal{O}, \mathcal{I}; \mathcal{J})$. If a_n is in the span modulo $\mathcal{P}\mathcal{O}_{\mathcal{P}}$ of $a'_1, \ldots, a'_{r_0}$, then there is a matrix $E_2 = \begin{pmatrix} I & * \\ & 1 \end{pmatrix}$ such that $(a'_1 \cdots a'_{n-1} a_n) E_2 = (a'_1 \cdots a'_n)$ where $a'_n \equiv 0 \pmod{\mathcal{P}\mathcal{O}_{\mathcal{P}}}$; note that $G_2 = \begin{pmatrix} E_2 & \\ & {}^tE_2^{-1} \end{pmatrix} \in \Gamma(\mathcal{O}, \ldots, \mathcal{O}, \mathcal{I}; \mathcal{J})$ and we take C_0 to be $G_1 G_2$. If $r_0 = n-1$ then we are now done, regardless of whether a_n is in the span (modulo $\mathcal{P}\mathcal{O}_{\mathcal{P}}$) of $a'_1, \ldots, a'_{r_0}$.

So suppose $r_0 < n-1$ (and thus $a'_{n-1} \equiv 0 \pmod{\mathcal{P}\mathcal{O}_\mathcal{P}}$) and a_n is not in the span (modulo $\mathcal{P}\mathcal{O}_\mathcal{P}$) of $a'_1, \dots, a'_{r_0-1}$.

Choose $\eta \in \mathcal{I}^{-1}, \rho \in \mathcal{I}$ such that $\eta \equiv \rho \equiv 1 \pmod{\mathcal{P}}$ and choose $\nu \in \mathcal{P}$ such that $(\nu, \eta\rho) = 1$. Thus there are $\alpha, \beta \in \mathcal{O}$ so that $\alpha\nu - \rho\eta\beta = 1$. Then with $E_3 = \left(\begin{smallmatrix} I & & \\ & \alpha & \eta\beta \\ & \rho & \nu \end{smallmatrix}\right)$, $(a'_1 \dots a'_{n-1} a_n)E_3 = (a'_1 \dots a'_{n-2} a''_{n-1} a''_n)$ with $a''_{n-1} \equiv a_n \pmod{\mathcal{P}}$, $a''_n \equiv 0 \pmod{\mathcal{P}}$. Note that $G_3 = \left(\begin{smallmatrix} E_3 & \\ & {}^tE_3^{-1} \end{smallmatrix}\right) \in \Gamma(\mathcal{O}, \dots, \mathcal{O}, \mathcal{I}; \mathcal{J})$. Let E_4 be the permutation matrix that permutes columns r_0 and $n-1$; then $G_4 = \left(\begin{smallmatrix} E_4 & \\ & {}^tE_4^{-1} \end{smallmatrix}\right) \in \Gamma(\mathcal{O}, \dots, \mathcal{O}, \mathcal{I}; \mathcal{J})$ and $(a_1 \dots a_n)G_1G_3G_4 \equiv (a'_1 \dots a'_{r_0-1} a_n 0 \dots 0) \pmod{\mathcal{P}}$. Hence in this case we take $C_0 = G_1G_3G_4$.

Thus there is a matrix C_0 and integer r_0 such that $\left(\begin{smallmatrix} C_0 & \\ & {}^tC_0^{-1} \end{smallmatrix}\right) \in \Gamma(\mathcal{O}, \dots, \mathcal{O}, \mathcal{I}; \mathcal{J})$ and $\Omega_0 = \Lambda C_0 \left(\begin{smallmatrix} \mathcal{P}I_{r_0} & \\ & I_{n-r_0} \end{smallmatrix}\right)$. Then with renewed notation, $M_j\left(\begin{smallmatrix} C_0 & \\ & {}^tC_0^{-1} \end{smallmatrix}\right)$ has the form $(a_1, \dots, a_n | b_1, \dots, b_n) = (A_0 A_1 | B)$, $A_1 \equiv 0 \pmod{\mathcal{P}}$.

Note that while C_0 is not uniquely determined, Ω_0 is.

Step 2. First note that [5, Lemma 7.2] easily generalizes to number fields, where we "permute" b_ℓ and b_n as we "permuted" a'_{r_0} and a_n in the preceding paragraph. Thus with

$$M_j \begin{pmatrix} C_0 & \\ & {}^tC_0^{-1} \end{pmatrix} = (A|B) = (a_1, \dots, a_n | b_1, \dots, b_n),$$

where $b_1, \dots, b_{r_0}$ are in the span mod $\mathcal{P}\mathcal{O}_\mathcal{P}$ of $a_1, \dots, a_{r_0}$, and the rank mod $\mathcal{P}\mathcal{O}_\mathcal{P}$ of $(a_1, \dots, a_{r_0}, b_{r_0+1}, \dots, b_n)$ is j. Thus for some $C = \left(\begin{smallmatrix} I_{r_0} & \\ & E \end{smallmatrix}\right)$ with $\left(\begin{smallmatrix} C & \\ & {}^tC^{-1} \end{smallmatrix}\right) \in \Gamma(\mathcal{O}, \dots, \mathcal{O}, \mathcal{I}; \mathcal{J})$, we have $(A|B)\left(\begin{smallmatrix} C & \\ & {}^tC^{-1} \end{smallmatrix}\right) = (A'|B')$ with $a'_i = a_i$ for $i \le r_0$, and j the rank mod $\mathcal{P}\mathcal{O}_\mathcal{P}$ of $(a_1, \dots, a_{r_0}, b'_{r_0+1}, \dots, b'_j)$. We want to accomplish the above rearrangement, as well as replacing $b'_{j+1}, \dots, b'_n$ with vectors in the span mod $\mathcal{P}\mathcal{O}_\mathcal{P}$ of $a_1, \dots, a_{r_0}$; we want to identify these modifications with a uniquely determined lattice.

From Step 1 we have $\Lambda = \Lambda_0 \oplus \Delta_1$, $\Omega_0 = \mathcal{P}\Lambda_0 \oplus \Delta_1$ with Δ_1 uniquely determined modulo $\mathcal{P}\Lambda$. This corresponds to a splitting $\Lambda^\# = \Lambda'_0 \oplus \Lambda'_1$ of the (formal) dual of Λ, where Λ'_0 is orthogonal to Δ_1 and Λ'_1 is orthogonal to Λ_0. (So

$$\Lambda^\# = \mathcal{O}_\mathcal{P} y_1 \oplus \dots \oplus \mathcal{O}_\mathcal{P} y_n$$

and the basis $\{y_1, \dots, y_n\}$ is dual to $\{x_1, \dots, x_n\}$.)

Let V be the $\mathcal{O}_\mathcal{P}/\mathcal{P}\mathcal{O}_\mathcal{P}$-space consisting of all $j \times 1$ matrices, and let U be the subspace spanned by $\overline{a}_1, \dots, \overline{a}_{r_0}$. Let

$$\Omega'_1 = \ker(\Lambda^\# \to \Lambda^\#(B) \bmod \mathcal{P}\mathcal{O}_\mathcal{P} \to V/U),$$

where $\Lambda^\# \to \Lambda^\#(B)$ corresponds to $y_i \mapsto b_i$, and the map into V/U is the canonical projection map. Thus as in Step 1, we can find a matrix $C = \left(\begin{smallmatrix} I_{r_0} & \\ & E \end{smallmatrix}\right)$ so that $\left(\begin{smallmatrix} C & \\ & {}^tC^{-1} \end{smallmatrix}\right) \in \Gamma(\mathcal{O}, \dots, \mathcal{O}, \mathcal{I}; \mathcal{J})$ and $(A|B)\left(\begin{smallmatrix} C & \\ & {}^tC^{-1} \end{smallmatrix}\right) = (A'|B')$ with $b'_0 = b_0, \dots, b'_{r_0} = b_{r_0}, b'_{j+1}, b'_n$ in the span mod $\mathcal{P}\mathcal{O}_\mathcal{P}$ of $a_0, \dots, a_{r_0}$. Also, setting

$C_1 = C_0 C$,

$$\Omega_1' = \Lambda^{\#}\, {}^tC_1^{-1} \begin{pmatrix} I_{r_0} & & \\ & \mathcal{P}I_{j-r_0} & \\ & & I_{n-j} \end{pmatrix};$$

$\Lambda^{\#} = \Lambda^{\#}\, {}^tC_1^{-1} = \Lambda_0' \oplus \Lambda_2' \oplus \Lambda_3'$ with rank $\Lambda_2' = j - r_0$ and $\Lambda_0' \oplus \Lambda_3'$ uniquely determined modulo $\mathcal{P}\Lambda^{\#}$. Correspondingly,

$$\Omega_0 = \Lambda C_1 \begin{pmatrix} \mathcal{P}I_{r_0} & \\ & I_{n-r_0} \end{pmatrix}.$$

Note that $\Lambda = \Lambda C_1 = \Lambda_0 \oplus \Delta_2 \oplus \Lambda_3$. Set

$$\Omega_1 = \Lambda C_1 \begin{pmatrix} \mathcal{P}I_{r_0} & & \\ & I_{j-r_0} & \\ & & \mathcal{P}I_{n-j} \end{pmatrix} = \mathcal{P}\Lambda_0 \oplus \Delta_2 \oplus \mathcal{P}\Lambda_3.$$

Since $\Lambda_0' \oplus \Lambda_3'$ are uniquely determined modulo $\mathcal{P}\Lambda^{\#}$, Δ_2 is uniquely determined modulo $\mathcal{P}\Lambda$.

Step 3. Write

$$M_j \begin{pmatrix} C_1 & \\ & {}^tC_1^{-1} \end{pmatrix} = (a_1 \cdots a_n | b_1 \cdots b_n) = (A_0 A_1 A_3 | B_0 B_1 B_3)$$

where j is the rank modulo $\mathcal{P}\mathcal{O}_{\mathcal{P}}$ of (A_0, B_1), $A_1, A_3 \equiv 0 \pmod{\mathcal{P}\mathcal{O}_{\mathcal{P}}}$, and B_0, B_3 are in the (column) span modulo $\mathcal{P}\mathcal{O}_{\mathcal{P}}$ of A_0. We want to modify A_1 to be of the form (A_1', A_2') where $A_2' \equiv 0 \pmod{\mathcal{P}^2\mathcal{O}_{\mathcal{P}}}$. Recall that we have

$$\Lambda = \Lambda_0 \oplus \Delta_2 \oplus \Lambda_3, \Omega_1 = \mathcal{P}\Lambda_0 \oplus \Delta_2 \oplus \mathcal{P}\Lambda_3,$$

with rank $\Lambda_0 = r_0$, rank $\Delta_2 = j - r_0$. Renewing our notation, let $(x_1, \ldots, x_n)$ be a basis corresponding to this decomposition of Λ.

Recall we have fixed $\mu \in \mathcal{P}^{-1} - \mathcal{O}$ so that μ is relatively prime to $\mathcal{I}\mathcal{J}\partial^{-1}$; set

$$\mathcal{P}\Omega_2 = \ker(\Omega_1 \to \Omega_1(\mu A) \bmod \mathcal{P}\mathcal{O}_{\mathcal{P}})$$

where $\Omega_1 \to \Omega_1(\mu A)$ denotes the map taking x_i to μa_i, so $\mu\Omega_1(A) \bmod \mathcal{P}\mathcal{O}_{\mathcal{P}}$ is spanned by $\overline{a}_1, \ldots, \overline{a}_{r_0}, \mu\overline{a}_{r_0+1}, \ldots, \mu\overline{a}_j$ with $\overline{a}_1, \ldots, \overline{a}_{r_0}$ linearly independent modulo $\mathcal{P}\mathcal{O}_{\mathcal{P}}$ (recall $a_i \simeq 0 \pmod{\mathcal{P}\mathcal{O}_{\mathcal{P}}}$ for $i > j$). Thus

$$\mathcal{P}\Omega_2 = \mathcal{P}^2\Lambda_0 \oplus \mathcal{P}\Lambda_1 \oplus \Lambda_2 \oplus \mathcal{P}\Lambda_3$$

where Λ_2 is uniquely determined modulo $\mathcal{P}\Omega_1$. As in the previous steps, we can find a matrix $C = \begin{pmatrix} I_{r_0} & & \\ & E & \\ & & I_{n-j} \end{pmatrix}$ such that $\begin{pmatrix} C & \\ & {}^tC^{-1} \end{pmatrix} \in \Gamma(\mathcal{O}, \ldots, \mathcal{O}, \mathcal{I}; \mathcal{J})$ and

$$\mathcal{P}\Omega_2 = \Lambda C_1 C \begin{pmatrix} \mathcal{P}^2 I_{r_0} & & & \\ & \mathcal{P}I_{r_1} & & \\ & & I_{r_2} & \\ & & & \mathcal{P}I_{n-j} \end{pmatrix}.$$

Correspondingly, $(A_0A_1A_3|B_0B_1B_3)\begin{pmatrix} C & \\ & {}^tC^{-1}\end{pmatrix} = (A_0A_1'A_2A_3|B_0B_1'B_2B_3)$ with $A_1' \equiv 0 \pmod{\mathcal{P}\mathcal{O}_\mathcal{P}}$, $A_2 \equiv 0 \pmod{\mathcal{P}^2\mathcal{O}_\mathcal{P}}$, and $a_1,\dots,a_{r_0},\mu a'_{r_0+1},\dots,\mu a'_{r_0+r_1}$ linearly independent modulo $\mathcal{P}\mathcal{O}_\mathcal{P}$ where $r_1 = \operatorname{rank}\Lambda_1$ (and so A_1' is $j\times r_1$). Let $C(\Omega,\Lambda_1) = C_2 = C_1C$.

Step 4. Write $M_j\begin{pmatrix} C_2 & \\ & {}^tC_2^{-1}\end{pmatrix} = (A_0,A_1,A_2,A_3|B_0,B_1,B_2,B_3)$. So $A_1, A_3 \equiv 0 \pmod{\mathcal{P}\mathcal{O}_\mathcal{P}}$, $A_2 \equiv 0 \pmod{\mathcal{P}^2\mathcal{O}_\mathcal{P}}$, and the columns of $(A_0,\mu A_1)$ are linearly independent modulo $\mathcal{P}\mathcal{O}_\mathcal{P}$. Also, B_0, B_3 are in the column span modulo $\mathcal{P}\mathcal{O}_\mathcal{P}$ of A_0.

Since B_0, B_1 are in $\operatorname{span}_\mathcal{P} A_0$, we can solve

$$A_0Y_0' \equiv -B_0 (\operatorname{mod} \mathcal{P}\mathcal{O}_\mathcal{P}), A_0Y_3 \equiv -B_3 (\operatorname{mod} \mathcal{P}\mathcal{O}_\mathcal{P}).$$

Note that as B^tA is symmetric and $A_1,A_2,A_3 \equiv 0(\operatorname{mod} \mathcal{P}\mathcal{O}_\mathcal{P})$, $B_0^tA_0$ is symmetric modulo $\mathcal{P}\mathcal{O}_\mathcal{P}$. Also, since A_0 has full rank modulo $\mathcal{P}$, there is some matrix $E \in GL_j(\mathcal{O}_\mathcal{P})$ such that $EA_0 = \begin{pmatrix} A' \\ 0 \end{pmatrix}$. Writing $EB_0 = \begin{pmatrix} B' \\ B'' \end{pmatrix}$, we see $B'' \equiv 0(\operatorname{mod} \mathcal{P}\mathcal{O}_\mathcal{P})$ since $E(B_0^tA_0)^tE$ is symmetric modulo $\mathcal{P}\mathcal{O}_\mathcal{P}$. Thus Y_0' is the unique solution modulo $\mathcal{P}$ to $A'Y_0' \equiv -B'(\operatorname{mod} \mathcal{P}\mathcal{O}_\mathcal{P})$; since $(A')^{-1}B'$ is symmetric modulo $\mathcal{P}\mathcal{O}_\mathcal{P}$, we can choose Y_0' to be symmetric.

Let

$$\begin{aligned}(A_0,A_1,A_2,A_3|B_0',B_1,B_2,B_3')\\ = (A_0,A_1,A_2,A_3|B_0,B_1,B_2,B_3)\begin{pmatrix} I &&&& Y_0 & 0 & 0 & Y_3 \\ & I &&& 0 &&& \\ && I && 0 &&& \\ &&& I & {}^tY_3 &&& \\ &&&& I &&& \\ &&&&& I && \\ &&&&&& I & \\ &&&&&&& I \end{pmatrix},\end{aligned}$$

$B_0', B_3' \equiv 0 \pmod{\mathcal{P}\mathcal{O}_\mathcal{P}}$. Then just as we argued about Y_0', there is a unique modulo $\mathcal{P}\mathcal{O}_\mathcal{P}$ symmetric solution Y' to

$$(A_0,\mu A_1)Y' \equiv -(\mu(B_0' + A_3\ {}^tY_3), B_1) \pmod{\mathcal{P}\mathcal{O}_\mathcal{P}}.$$

Decompose Y' as $\begin{pmatrix} Y_0'' & Y_2 \\ {}^tY_2 & Y_1\end{pmatrix}$; choose $\delta \in \mathcal{P}$ so that $\delta\mu \equiv 1 \pmod{\mathcal{P}}$ and set $Y_0 = Y_0' + \delta Y_0''$.

Note that since $\operatorname{rank}_\mathcal{P}(A_0,B_1) = r_0 + r_1$, we have $\operatorname{rank}_\mathcal{P}(B_1 + A_0Y_0) = \operatorname{rank}_\mathcal{P} B_1 = r_1$. Since $-\mu A_1Y_1 \equiv B_1 + A_0Y_2 \pmod{\mathcal{P}\mathcal{O}_\mathcal{P}}$, we must have $\det Y_1 \in \mathcal{O}_\mathcal{P}^\times$.

Take $Y = \begin{pmatrix} W & 0 & W_3 \\ 0 & I & \\ {}^tW_3 && \end{pmatrix} = \begin{pmatrix} W_0 & W_2 & 0 & W_3 \\ {}^tW_2 & \mu W_1 && \\ 0 && I & \\ {}^tW_3 &&& \end{pmatrix}$ to be a symmetric matrix in $(\mathcal{I}_i\mathcal{I}_j\mathcal{J}\partial^{-1})$ with $W_0 \equiv Y_0 \pmod{\mathcal{P}^2\mathcal{O}_\mathcal{P}}$ and $W_i \equiv Y_i \pmod{\mathcal{P}\mathcal{O}_\mathcal{P}}$ for $i = 1,2,3$.

Then

$$(A_0, A_1, A_2, A_3 | B_0, B_1, B_2, B_3) \begin{pmatrix} I & Y \\ 0 & I \end{pmatrix} = (A_0, A_1, A_2, A_3 | B_0'', B_1', B_2', B_3')$$

with $B_0'' \equiv 0 \pmod{\mathcal{P}^2\mathcal{O}_\mathcal{P}}$, $B_2' \equiv B_2 \pmod{\mathcal{P}^2\mathcal{O}_\mathcal{P}}$, $B_1', B_3' \equiv 0 \pmod{\mathcal{P}\mathcal{O}_\mathcal{P}}$.

Let $C = C(\Omega, \Lambda_1)$. Also, identifying $S_i(\mathcal{P})$ with a matrix giving its action, let

$$S^{(j)}(\Omega) = \left(\prod_{i=r_0+1}^{r_0+r_1} S_i(\mathcal{P})\right)\left(\prod_{i=r_0+r_1+1}^{j} S_i(\mathcal{P}^2)\right).$$

We see that, with renewed notation,

$$M_j \begin{pmatrix} C & \\ & {}^tC^{-1} \end{pmatrix} \begin{pmatrix} I & Y \\ & I \end{pmatrix} (S^{(j)}(\Omega))^{-1} = (A_0, A_1, A_2, A_3 | B_0, B_1, B_2, B_3)$$

with $A_3, B_3 \equiv 0 \pmod{\mathcal{P}\mathcal{O}_\mathcal{P}}$, $B_0, B_1, B_2 \equiv 0$ $(\mathcal{P}^2\mathcal{O}_\mathcal{P})$.

However, while the matrices for $S_i(\mathcal{P}^{-2})$ lie in Γ, the matrices for $S_i(\mathcal{P}^{-1})$ do not. We remedy this as follows.

For $r_0 < i \le r_0 + r_1$, choose $\alpha_i \in \mathcal{P}^{-1}$, $\beta_i \in \mathcal{P}\mathcal{I}_i^2\mathcal{J}\partial^{-1}$, $\gamma_i \in \mathcal{P}^{-1}\mathcal{I}_i^{-2}\mathcal{J}^{-1}\partial$ so that $\alpha_i\delta - \beta_i\gamma_i = 1$ and for any prime $\mathcal{Q} \neq \mathcal{P}$ dividing δ, $\mathcal{Q}$ does not divide β_i. (Recall that our choice of δ ensures $\delta \in \mathcal{P} - \mathcal{P}^2$.) Set $\underline{\alpha} = \mathrm{diag}(\dots, \alpha_i, \dots)$, an $r_1 \times r_1$ matrix; define $\underline{\beta}, \underline{\gamma}$ in an analogous fashion. So

$$\begin{pmatrix} I_{r_0} & & & 0_{r_0} & & \\ & \underline{\alpha} & & & \underline{\beta} & \\ & & I & & & 0 \\ 0_{r_0} & & & I_{r_0} & & \\ & \underline{\gamma} & & & \delta I & \\ & & 0 & & & I \end{pmatrix}$$

gives the action of $\prod_{i=r_0+1}^{r_0+r_1} S_i(\mathcal{P}^{-1})$.

Now consider

$$\begin{pmatrix} I & -\mu W_1 \\ & I \end{pmatrix} \begin{pmatrix} \underline{\alpha} & \underline{\beta} \\ \underline{\gamma} & \delta I \end{pmatrix} \equiv \begin{pmatrix} \underline{\alpha} - \mu W_1 \underline{\gamma} & \underline{\beta} - W_1 \\ \underline{\gamma} & \delta I \end{pmatrix} \pmod{\mathcal{P}}.$$

We find that $\left(\begin{smallmatrix} \underline{\beta} & -W_1 \\ & \delta I \end{smallmatrix}\right)$ is a coprime symmetric right-hand pair for $\Gamma(\mathcal{I}_{r_0+1}, \dots, \mathcal{I}_{r_0+r_1'}; \mathcal{J})$ $(2r_1' \times 2r_1'$ matrices). Thus by Lemma 6.1, there exist matrices U, V so that

$$\begin{pmatrix} U & \underline{\beta} + \delta W_1 \\ V & \delta I \end{pmatrix} \in \Gamma(\mathcal{I}_{r_0+1}, \dots, \mathcal{I}_{r_0+r_1'}; \mathcal{J}).$$

Hence

$$X^{-1} = \begin{pmatrix} I_{r_0} & & & 0_{r_0} & & \\ & U & & & \underline{\beta} - W_1 & \\ & & I & & & 0 \\ 0_{r_0} & & & I_{r_0} & & \\ & V & & & \delta I & \\ & & 0 & & & I \end{pmatrix} \in \Gamma,$$

and with $Y' = \begin{pmatrix} w_1 & w_2 & w_3' \\ {}^t w_2 & & \\ {}^t w_3' & & \end{pmatrix}$ and

$$N^{-1} = \begin{pmatrix} C & \\ & {}^t C^{-1} \end{pmatrix} \begin{pmatrix} I & -Y' \\ & I \end{pmatrix} X^{-1} \left(\prod_{i=r_0+r_1'+1}^{j} S_i(\mathcal{P}^{-2}) \right),$$

we have $M_j N^{-1} = (A_0, A_1, A_2, A_3 | B_0, B_1, B_2, B_3)$ with $A_3, B_3 \equiv 0 \pmod{\mathcal{P}\mathcal{O}_{\mathcal{P}}}$, $B_0, B_1, B_2 \equiv 0 \pmod{\mathcal{P}^2\mathcal{O}_{\mathcal{P}}}$. Thus by an easy generalization of [5, Lemma 7.1], $MN^{-1} \in \Gamma \cap \Gamma'$. Also, since

$$\begin{pmatrix} \delta I & -\underline{\beta} + W_1 \\ -\underline{\gamma} & \underline{\alpha} \end{pmatrix} \begin{pmatrix} U & \underline{\beta} - W_1 \\ V & \delta I \end{pmatrix} = \begin{pmatrix} I & 0 \\ V' & I \end{pmatrix} \in \Gamma(\mathcal{I}'_{r_0+1}, \ldots, \mathcal{I}'_{r_0+r_1'}; \mathcal{J}),$$

we have

$$f'|N = f'|S^{(j)}(\Omega) \begin{pmatrix} I & -Y \\ & I \end{pmatrix} \begin{pmatrix} C^{-1} & \\ & {}^t C \end{pmatrix}. \qquad \square$$

Proposition 4.3. *For $f' \in \mathcal{M}_k(\Gamma(\mathcal{I}_1, \ldots, \mathcal{I}_n; \mathcal{P}\mathcal{J}))$, we have*

$$f'|T(\mathcal{P}) = N(\mathcal{P})^{n(k-n-1)/2} \sum_{\Omega, Y_0} f'|S(\Omega) \begin{pmatrix} I & & Y_0 & \\ & I & & 0 \\ & & I & \\ & & & I \end{pmatrix} \begin{pmatrix} C^{-1} & \\ & {}^t C \end{pmatrix}$$

and $Y_0 \in (\mathcal{I}_i \mathcal{I}_\ell \mathcal{J} \partial^{-1})$ varies over symmetric $r \times r$ matrices modulo $\mathcal{P}$. Here $r =$ $\mathrm{mult}_{\{\Lambda:\Omega\}}(\mathcal{P})$, $\Omega = \Lambda C \mathrm{diag}(\mathcal{P} I_r, I_{n-r})$, and $S(\Omega) = \prod_{i=r+1}^{n} S_i(\mathcal{P})$.

Proof. To find coset representatives for $T(\mathcal{P})$, take $M \in \Gamma(\mathcal{I}_1, \ldots, \mathcal{I}_n; \mathcal{J})$; write $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Let $\Omega = \ker(\Lambda \mapsto \Lambda(A) \pmod{\mathcal{P}})$, and choose $C = C(\Omega)$ so that $\Omega = \Lambda C \begin{pmatrix} \mathcal{P} I_r & \\ & I_{n-r} \end{pmatrix}$. Thus $(A|B)C = (A_0 A_1 | B_0 B_1)$, $A_1 \equiv 0 \pmod{\mathcal{P}\mathcal{O}_{\mathcal{P}}}$, $\mathrm{rank}_{\mathcal{P}} A_0 = r$ where A_0 is $n \times r$, and $B_0 \in \mathrm{span}_{\mathcal{P}} A_0$. Choose symmetric Y_0 with i, ℓ-entry in $\mathcal{I}_i \mathcal{I}_\ell \mathcal{J} \partial^{-1}$ such that $A_0 Y_0 \equiv B_0 \pmod{\mathcal{P}\mathcal{O}_{\mathcal{P}}}$. Then with $Y = \begin{pmatrix} Y_0 & \\ & \mu I \end{pmatrix}$, $(A|B)\begin{pmatrix} C & \\ & {}^t C^{-1} \end{pmatrix}\begin{pmatrix} I & -Y \\ & I \end{pmatrix} S^{-1}(\Omega) = (A'|B')$ where $S(\Omega) = \prod_{i=r+1}^{n} S_i(\mathcal{P})$ and $B' \equiv 0 \pmod{\mathcal{P}\mathcal{O}_{\mathcal{P}}}$. As before, choose diagonal $(n-r) \times (n-r)$ matrices $\underline{\alpha}$, $\underline{\beta}$, $\underline{\gamma}$ so that the action of $S(\Omega)$ is given by $\begin{pmatrix} I_r & & 0_r & \\ & \underline{\alpha} & & \underline{\beta} \\ 0_r & & I_r & \\ & \underline{\gamma} & & \delta I \end{pmatrix}$. Then

$$\begin{pmatrix} I & -Y \\ & I \end{pmatrix} \begin{pmatrix} I & & 0 & \\ & \underline{\alpha} & & \underline{\beta} \\ 0 & & I & \\ & \underline{\gamma} & & \delta I \end{pmatrix} = \begin{pmatrix} I & & -Y & \\ & I & & 0 \\ & & I & \\ & & & I \end{pmatrix} \begin{pmatrix} I & & 0 & \\ & \underline{\alpha} + \mu\underline{\gamma} & & \underline{\beta} + \delta I \\ 0 & & I & \\ & \underline{\gamma} & & \delta I \end{pmatrix}.$$

Here $\underline{\beta} \equiv 0 \pmod{\mathcal{P}\mathcal{O}_\mathcal{P}}$, $\mathrm{rank}_\mathcal{P}(\underline{\beta} + \mu\delta I) = n - r$. Thus $\left(\begin{smallmatrix} \underline{\beta} - \mu\delta I \\ \delta I \end{smallmatrix} \right)$ is a symmetric coprime right-hand pair for $\Gamma(\mathcal{I}_{r+1}, \ldots, \mathcal{I}_n; \mathcal{J})$, hence there are U, V so that $\begin{pmatrix} I & & 0 & \\ & U & & \underline{\beta} + \mu\delta I \\ 0 & & I & \\ & V & & \delta I \end{pmatrix} \in \Gamma(\mathcal{I}_1, \ldots, \mathcal{I}_n; \mathcal{J})$. Since $\left(\begin{smallmatrix} U & \underline{\beta} + \mu\delta I \\ V & \delta I \end{smallmatrix} \right) \left(\begin{smallmatrix} \delta I & -\underline{\beta} \\ -\underline{\gamma} & \underline{\alpha} \end{smallmatrix} \right) \in \Gamma(\mathcal{I}_{r+1}, \ldots, \mathcal{I}_n; \mathcal{P}\mathcal{J})$ we get the result as claimed. $\square$

5. Evaluating the Action of the Hecke Operators

When evaluating the action of the operators $T_j(\mathcal{P}^2)$, we encounter incomplete character sums. To complete these, we define modified operators as follows.

Definition. For $\mathcal{P}$ a prime ideal and $1 \le j \le n$, define

$$\widetilde{T}_j(\mathcal{P}^2) = N(\mathcal{P})^{j(k-n-1)} \sum_{0 \le \ell \le j} \beta(n-\ell, j-\ell) S_{\ell+1}(\mathcal{P}) \cdots S_j(\mathcal{P}) T_\ell(\mathcal{P}^2).$$

We will also need the following rather technical result.

Proposition 5.1. *Let T_1 be a symmetric $r_1 \times r_1$ matrix whose i, ℓ-entry lies in $(\mathcal{I}_{r_0+i}\mathcal{I}_{r_0+\ell}\mathcal{J})^{-1}$, and whose ith diagonal entry lies in $2\mathcal{I}_{r_0+i}^{-2}\mathcal{J}^{-1}$; fix $\mu \in \mathcal{P}^{-1} - \mathcal{O}$. With W varying over all symmetric $r_1 \times r_1$ matrices modulo $\mathcal{P}$ with i, ℓ-entry in $\mathcal{I}_{r_0+i}\mathcal{I}_{r_0+\ell}\mathcal{J}\partial^{-1}$,*

$$\sum_W \mathrm{e}\{\mu T_1 W\} = \sum_{0 \le m \le r_1'} \sum_{\Delta, U} \mathrm{e}\{\mu T_\Delta U\}$$

where for each m, $\overline{\Delta}$ varies over dimension m subspaces of $\overline{\Lambda}_1$, $\overline{\Delta} \simeq T_\Delta \pmod{\mathcal{P}}$, and U varies over all $m \times m$ symmetric matrices modulo $\mathcal{P}$.

Proof. For a moment, let's fix W. Since W is symmetric, we can view it as the matrix of a quadratic form on an r_1' dimensional $\mathcal{O}/\mathcal{P}$ space $V = L/\mathcal{P}L$, $L = (\mathcal{I}_{r_0+1}^{-1} y_1 \oplus \cdots \oplus \mathcal{I}_{r_0+r_1} y_{r_1})^{\mathcal{J}^{-1}\partial}$. (When $\mathcal{P}$ is dyadic, let W define an integral quadratic form on

$$L = (\mathcal{O}_\mathcal{P}\mathcal{I}_{r_0+1}^{-1} y_1 \oplus \cdots \oplus \mathcal{O}_\mathcal{P}\mathcal{I}_{r_0+r_1}^{-1} y_{r_1})^{\mathcal{J}^{-1}\partial},$$

and let $V = L/\mathcal{P}L$, a quadratic space over $\mathcal{O}_\mathcal{P}/\mathcal{P}\mathcal{O}_\mathcal{P} \approx \mathcal{O}/\mathcal{P}$. We use [6, Sec. 93] to understand the structure of L and thereby of V.) The radical of this space is uniquely defined, so for some $G \in GL_{r_1}(\mathcal{O})$,

$${}^tG^{-1} W G^{-1} \equiv \begin{pmatrix} U & \\ & 0 \end{pmatrix} \pmod{\mathcal{P}}$$

where U is $m\times m$ with $\operatorname{rank}_{\mathcal{P}} U=m$. (So $VG^{-1}=J\oplus \operatorname{rad} V$ where J is a regular space whose isometry class is uniquely determined by V, and $J\simeq U$.)

So

$$\begin{aligned}\mathrm{e}\{\alpha T_1 W\} &= \mathrm{e}\left\{\alpha T_1 {}^tG\begin{pmatrix}U & \\ & 0\end{pmatrix}G\right\}\\ &= \mathrm{e}\left\{\alpha(GT_1{}^tG)\begin{pmatrix}U & \\ & 0\end{pmatrix}\right\}\\ &= \mathrm{e}\{\alpha SU\}\end{aligned}$$

where $GT_1\,{}^tG=\left(\begin{smallmatrix}S & *\\ * & *\end{smallmatrix}\right)$, S an $m\times m$ matrix. Here we take Λ to be a rank n lattice as in the previous section, and we equip Λ with a quadratic form such that $\Lambda\simeq T$; thus with Λ_1 as in the previous section, we have $\Lambda_1\simeq T_1$, and $\Delta=\Lambda_1{}^tG\left(\begin{smallmatrix}I_m\\ 0\end{smallmatrix}\right)\simeq S$. So S corresponds to an m-dimensional subspace $\overline{\Delta}$ of the $\mathcal{O}/\mathcal{P}$-space $\overline{\Lambda}_1$. Thus each W gives rise to (at least one) pair $(\overline{\Delta},U)$, $\overline{\Delta}$ an m-dimensional subspace of $\overline{\Lambda}_1$, U an $m\times m$ integral symmetric matrix of rank m modulo $\mathcal{P}$.

With T_1 still fixed, fix m, $0\le m\le r_1$. We now define a map φ from all pairs $(\overline{\Delta},U)$ as above to symmetric $r_1\times r_1$ matrices W. Here $\overline{\Delta}$ is an m-dimensional subspace of $\overline{\Lambda}_1$, and U is an integral symmetric $m\times m$ matrix with $\operatorname{rank}_{\mathcal{P}} U=m$. For each such $\overline{\Delta}$ we fix some $G=G_\Delta\in GL_{r_1}(\mathcal{O})$ so that $\overline{\Delta}=\overline{\Lambda}_1{}^tG\left(\begin{smallmatrix}I_m\\ 0\end{smallmatrix}\right)$. We define $\varphi(\overline{\Delta},U)={}^tG\left(\begin{smallmatrix}U & \\ & 0\end{smallmatrix}\right)G$.

We first show that the image of φ consists of all symmetric $r_1\times r_1$ matrices W modulo $\mathcal{P}$ with $\operatorname{rank}_{\mathcal{P}} W=m$. Then we show that φ is injective.

As shown above, given any W in the codomain of φ,

$$W\equiv {}^tG\begin{pmatrix}U & \\ & 0\end{pmatrix}G \pmod{\mathcal{P}}$$

where U is $m\times m$, $m=\operatorname{rank}_{\mathcal{P}}U$, and $G\in GL_{r_1}(\mathcal{O})$.

Take $\overline{\Delta}=\overline{\Lambda}_1{}^tG\left(\begin{smallmatrix}I_m\\ 0\end{smallmatrix}\right)$. So $\overline{\Delta}$ is an m-dimensional subspace of $\overline{\Lambda}_1$, and thus ${}^tG\left(\begin{smallmatrix}I_m\\ 0\end{smallmatrix}\right)$ and ${}^tG_\Delta\left(\begin{smallmatrix}I_m\\ 0\end{smallmatrix}\right)$ each map a basis for $\overline{\Lambda}_1$ to a basis for $\overline{\Delta}$. Hence with $(x_1,\dots,x_{r_1})$ a basis for $\overline{\Lambda}_1$, $(x_1,\dots,x_{r_1}){}^tG=(y_1,\dots,y_{r_1})$, $(x_1,\dots,x_{r_1}){}^tG_\Delta=(z_1,\dots,z_{r_1})$, we must have $(y_1,\dots,y_m)\equiv(z_1,\dots,z_m){}^tC \pmod{\mathcal{P}}$ for some $C\in GL_m(\mathcal{O})$. Thus $(x_1,\dots,x_{r_1}){}^tG=(x_1,\dots,x_{r_1}){}^tG_\Delta\left(\begin{smallmatrix}{}^tC & *\\ 0 & *\end{smallmatrix}\right)$, meaning ${}^tG={}^tG_\Delta\left(\begin{smallmatrix}{}^tC & *\\ 0 & *\end{smallmatrix}\right)$.

Hence modulo $\mathcal{P}$,

$$W\equiv\ {}^tG\begin{pmatrix}U & \\ & 0\end{pmatrix}G=\ {}^tG_\Delta\begin{pmatrix}{}^tC & *\\ 0 & *\end{pmatrix}\begin{pmatrix}U & \\ & 0\end{pmatrix}\begin{pmatrix}C & 0\\ * & *\end{pmatrix}G_\Delta=\varphi(\overline{\Delta},\ {}^tCUC).$$

Thus φ is surjective.

Now we show φ is injective. Say

$$W\equiv\varphi(\Delta_1,U_1)\equiv\varphi(\Delta_2,U_2) \pmod{\mathcal{P}}.$$

Thus with $G_i=G_{\Delta_i}$, we have

$$W\equiv {}^tG_1\begin{pmatrix}U_1 & \\ & 0\end{pmatrix}G_1\equiv {}^tG_2\begin{pmatrix}U_2 & \\ & 0\end{pmatrix}G_2 \pmod{\mathcal{P}}.$$

So with $G = G_2 G_1^{-1}$,

$$\begin{pmatrix} U_1 & \\ & 0 \end{pmatrix} \equiv {}^tG \begin{pmatrix} U_2 & \\ & 0 \end{pmatrix} G \pmod{\mathcal{P}}.$$

Since the columns of U_i are linearly independent modulo $\mathcal{P}$, we must have $G \equiv \left(\begin{smallmatrix} C & 0 \\ * & * \end{smallmatrix}\right) \pmod{\mathcal{P}}$.

Now we compare Δ_1, Δ_2; we will find that $\overline{\Delta}_1 = \overline{\Delta}_2$, so $G_1 = G_2$ and hence $U_1 \equiv U_2 \pmod{\mathcal{P}}$. With notation as before, we have

$$\begin{aligned}
(y_1, \ldots, y_m) &= (x_1, \ldots, x_{r_1'})\,{}^tG_1 \begin{pmatrix} I_m \\ 0 \end{pmatrix}, \\
(z_1, \ldots, z_m) &= (x_1, \ldots, x_{r_1'})\,{}^tG_2 \begin{pmatrix} I_m \\ 0 \end{pmatrix} \\
&= (x_1, \ldots, x_{r_1'})\,{}^tG_1\,{}^tG \begin{pmatrix} I_m \\ 0 \end{pmatrix}.
\end{aligned}$$

Thus given our knowledge of G, we see that $(y_1, \ldots, y_m) = (z_1, \ldots, z_m)\,{}^tC$, and hence $\overline{\Delta}_1 = \overline{\Delta}_2$. Thus $G_1 = G_2$, and consequently $U_1 \equiv U_2 \pmod{\mathcal{P}}$. Therefore φ is injective. □

We can now prove our main result. In the remark following the proof we demonstrate how to compute the geometric term $\alpha_j(\Omega, \Lambda)$.

Theorem 5.2. *Let $F \in \mathcal{M}_k(\chi)$ where χ is a character of the ideal class group and $\mathcal{M}_k(\chi)$ is as defined in Proposition* 3.2.

(1) *The Λ^Jth coefficient of $F|\widetilde{T}_j(\mathcal{P}^2)$ is*

$$\sum_{\mathcal{P}\Lambda \subseteq \Omega \subseteq \mathcal{P}^{-1}\Lambda} N(\mathcal{P})^{E_j(\Lambda,\Omega)} \chi(\mathcal{P})^{e_j(\Lambda,\Omega)} \alpha_j(\Omega, \Lambda) c_F(\Omega^{\mathcal{J}})$$

where $E_j(\Lambda, \Omega) = k(r_2 - r_0 + j) + r_0(r_0 + m_1 + 1) + r_1(r_1 + 1)/2 - j(n+1)$, $e_j(\Lambda, \Omega) = 2r_2 + r_1 = r_2 - r_0 + j$, and $\alpha_j(\Omega, \Lambda)$ is the number of totally isotropic codimension $n - j$ subspaces of $\Omega \cap \Lambda / \mathcal{P}(\Omega + \Lambda)$. Here $r_0 = \mathrm{mult}_{\{\Lambda:\Omega\}}(\mathcal{P})$, $m_1 = \mathrm{mult}_{\{\Lambda:\Omega\}}(\mathcal{O})$, $r_1 = m_1 - n + j$, and $r_2 = \mathrm{mult}_{\{\Lambda:\Omega\}}(\mathcal{P}^{-1})$.

(2) *The $\Lambda^{\mathcal{J}}$th coefficient of $F|T(\mathcal{P})$ is*

$$\sum_{\mathcal{P}\Lambda \subseteq \Omega \subseteq \Lambda} N(\mathcal{P})^{E(\Omega,\Lambda)} \chi(\mathcal{P})^{n-r} c_F(\Omega^{\mathcal{J}\mathcal{P}^{-1}})$$

where $r = \mathrm{mult}_{\{\Lambda:\Omega\}}(\mathcal{O})$ and $E(\Omega, \Lambda) = k(n - r) + r(r + 1)/2 - n(n + 1)/2$.

Proof. Take fractional ideals $\mathcal{I}_1, \ldots, \mathcal{I}_n, \mathcal{J}$ and $f' \in \mathcal{M}_k(\Gamma(\mathcal{I}_1', \ldots, \mathcal{I}_n'; \mathcal{J}))$,

$$\mathcal{I}_i' = \begin{cases} \mathcal{P}\mathcal{I}_i & \text{if } i \le j, \\ \mathcal{I}_i & \text{if } i > j. \end{cases}$$

In the preceding proposition, consider the subsum where we fix a choice of Ω:

$$\sum_{\Lambda_1,Y} f'|S(\Omega)\begin{pmatrix} I & Y \\ & I \end{pmatrix}\begin{pmatrix} C^{-1} & \\ & {}^tC \end{pmatrix} = \chi(\mathcal{P})^{2r_2+r_1}\sum_{\Lambda_1,Y} f''|\begin{pmatrix} I & Y \\ & I \end{pmatrix}\begin{pmatrix} C^{-1} & \\ & {}^tC \end{pmatrix}$$

where f'' is the component of F corresponding to the group $\Gamma(\mathcal{I}_1'',\dots,\mathcal{I}_n'';\mathcal{J})$,

$$\mathcal{I}_i'' = \begin{cases} \mathcal{P}\mathcal{I}_i & \text{if } 1 \le i \le r_0, \\ \mathcal{P}^{-1}\mathcal{I}_i & \text{if } r_0+r_1 < i \le j, \\ \mathcal{I}_i & \text{otherwise.} \end{cases}$$

Set $m_1 = r_1 + n - j$. Expanding f'' as a Fourier series supported on even $T \in ((\mathcal{I}_i''\mathcal{I}_\ell''\mathcal{J})^{-1})$, we find that for fixed Λ_1,

$$\begin{aligned} &\sum_Y f''|\begin{pmatrix} I & Y \\ & I \end{pmatrix}\begin{pmatrix} C^{-1} & \\ & {}^tC \end{pmatrix}(\tau) \\ &\quad = \sum_T c_{f''}(T)\mathrm{e}\{TC^{-1}\tau\, {}^tC^{-1}\}\sum_Y \mathrm{e}\{TY\} \\ &\quad = \sum_T c_{f''}(T)\mathrm{e}\{TC^{-1}\tau\, {}^tC^{-1}\} \\ &\quad\quad \times \sum_{W_0,W_1,W_2,W_3} \mathrm{e}\{T_0W_0\}\ \mathrm{e}\{\mu T_1W_1\}\ \mathrm{e}\{T_2W_2\}\ \mathrm{e}\{T_3W_3\} \end{aligned}$$

where $T = \begin{pmatrix} T_0 & T_2 & * & T_3 \\ {}^tT_2 & T_1 & * & * \\ * & * & * & * \\ {}^tT_3 & * & * & * \end{pmatrix}$. T_0 and W_0 are symmetric $r_0 \times r_0$ matrices with T_0 even, the i,ℓ-entry of W_0 in $\mathcal{I}_i\mathcal{I}_\ell\mathcal{J}\partial^{-1}$. Thus the sum on W_0 (T_0 fixed) is a complete character sum, yielding $N(\mathcal{P})^{r_0(r_0+1)}$ if $T_0 \equiv 0 \pmod{\mathcal{P}}$, and 0 otherwise. Similarly, the sums on W_2, W_3 are complete character sums. So

$$\sum_Y \mathrm{e}\{TY\} = \begin{cases} N(\mathcal{P})^{r_0(r_0+m_1+1)}\displaystyle\sum_{W_1}\mathrm{e}\{\mu T_1W_1\} & \text{if } T \in ((\mathcal{I}_i\mathcal{I}_\ell\mathcal{J})^{-1}), \\ 0 & \text{otherwise.} \end{cases}$$

With $(B(x_i,x_\ell)) = {}^tC^{-1}TC^{-1}$, take $\Lambda = \mathcal{I}_1x_1 \oplus \cdots \oplus \mathcal{I}_nx_n$. Then the sum on W_0, W_2, W_3 is nonzero if and only if $\Lambda^{\mathcal{J}}$ is even integral.

Let $(y_1 \dots y_n) = (x_1 \dots x_n)C$ and set $\Omega = \mathcal{I}_1''y_1 \oplus \cdots \oplus \mathcal{I}_n''y_n$. Then $(B(y_i,y_\ell)) = T$ and $c_{f''}(T)N(\mathcal{I}_1''\cdots\mathcal{I}_n'')^kN(\mathcal{J})^{nk/2} = c_F(\Omega^{\mathcal{J}})$.

Note that when $c_{f|S^\ell(\mathcal{P})}(T)$ is contributing to the $\Lambda^{\mathcal{J}}$th coefficient of $f|T_j(\mathcal{P}^2) \in \mathcal{M}_k(\Gamma(\mathcal{I}_1,\dots,\mathcal{I}_n;\mathcal{J}))$, it gets normalized by $N(\mathcal{I}_1\cdots\mathcal{I}_n)^kN(\mathcal{J})^{nk/2}$; when it is determining a coefficient of $f|S^\ell(\mathcal{P}) \in \mathcal{M}_k(\Gamma')$, $\Gamma' \simeq \Gamma(\mathcal{P}^{-\ell}\mathcal{I}_1,\mathcal{I}_2,\dots,\mathcal{I}_n;\mathcal{J})$, it gets normalized by $N(\mathcal{P})^{-\ell k}N(\mathcal{I}_1\dots\mathcal{I}_n)^kN(\mathcal{J})^{nk/2}$.

So the contribution from f'' to the $\Lambda\mathcal{J}$th Fourier coefficient of $f'|T_j(\mathcal{P}^2)$ is

$$N(\mathcal{P})^{j(k-n-1)+r_0(r_0+m_1+1)}\chi(\mathcal{P})^{2r_2+r_1'}\sum_{\Omega,\Lambda_1}N(\mathcal{P})^{k(2r_2+r_1)}c_F(\Omega^{\mathcal{J}})\sum_{W_1}\mathrm{e}\{\mu T_{\Lambda_1}W_1\}$$

where Ω varies subject to $\mathcal{P}\Lambda \subseteq \Omega \subseteq \mathcal{P}^{-1}\Lambda$, $\mathrm{mult}_{\{\Lambda:\Omega\}}(\mathcal{P}^{-1}) = r_0$, $\mathrm{mult}_{\{\Lambda:\Omega\}}(\mathcal{P}) = r_2$, $\overline{\Lambda}_1$ a codimension $n-j$ subspace of $\overline{\Omega}_1 \simeq \Omega\cap\Lambda/\mathcal{P}(\Omega+\Lambda)$, and W_1 varies modulo

$\mathcal{P}$ with i,ℓ-entry in $\mathcal{I}_i\mathcal{I}_\ell\mathcal{J}\partial^{-1}$, $\mathcal{P} \nmid \det W_1$. Here $T_{\Lambda_1} = (B(x_{r_0+i}, x_{r_0+\ell}))$ is $r_1 \times r_1$ where $\Lambda_1 = \mathcal{I}_{r_0+1}x_{r_0+1} \oplus \cdots \oplus \mathcal{I}_{r_0+r_1}x_{r_0+r_1}$. If T_1' is also a matrix associated to Λ_1 then there is a change of basis matrix G whose i,ℓ-entry lies in $\mathcal{I}_i\mathcal{I}_\ell^{-1}$ so that ${}^tGT_1'G = T_{\Lambda_1}$. Thus $\mathrm{e}\{\mu T_1' W_1\} = \mathrm{e}\{\mu T_{\Lambda_1}(GW_1{}^tG)\}$; as W_1 varies over invertible matrices modulo $\mathcal{P}$, so does $GW_1{}^tG$. Thus the sum on W_1 is independent of the choice of matrix associated to Λ_1.

We complete the character sum on W_1 by replacing $T_j(\mathcal{P}^2)$ with $\widetilde{T}_j(\mathcal{P}^2)$, then apply Proposition 5.1, where we consider $\sum_W \mathrm{e}\{\alpha T_1 W\}$ with W varying over all symmetric $r_1 \times r_1$ integral matrices modulo $\mathcal{P}$.

Notice that for $0 \le \ell \le j$,

$$S_{\ell+1}(\mathcal{P}) \cdots S_j(\mathcal{P})T_\ell(\mathcal{P}^2) : \mathcal{M}_k(\Gamma(\mathcal{I}_1', \ldots, \mathcal{I}_n'; \mathcal{J})) \to \mathcal{M}_k(\Gamma(\mathcal{I}_1, \ldots, \mathcal{I}_n; \mathcal{J})).$$

Also notice that the number of dimension $r_1 = m_1 - n + j$ lattices $\overline{\Lambda}_1$ containing some dimension $m_1 - n + \ell$ lattice $\overline{\Delta}$ is the number of ways to extend $\overline{\Delta}$ to a j-dimensional subspace of $\overline{\Omega}_1$ (where $\Omega = \Omega_0 \oplus \Omega_1 \oplus \Omega_2$, $\Lambda = \mathcal{P}\Omega_0 \oplus \Omega_1 \oplus \mathcal{P}^{-1}\Omega_2$). Extending $\overline{\Delta}$ is equivalent to choosing a $j-\ell$ dimensional subspace of an $n-\ell$ space (here $\dim \overline{\Omega}_1 = m_1$). So the number of $\overline{\Lambda}_1$ containing $\overline{\Delta}$ is $\beta(n-\ell, j-\ell) = \beta_{\mathcal{P}}(n-\ell, j-\ell)$.

Note that the coefficient of $f'|\widetilde{T}_j(\mathcal{P}^2)$ associated to $\Lambda^{\mathcal{J}} = (\mathcal{I}_1x_1 \oplus \cdots \oplus \mathcal{I}_nx_n)^{\mathcal{J}}$ carries a normalizing factor of $N(\mathcal{I}_1 \cdots \mathcal{I}_n)^k N(\mathcal{J})^{nk/2}$, while the coefficient of $f'|S^{(j)}(\mathcal{P})$ associated to $\Omega^{\mathcal{J}} = (\mathcal{I}_1''y_1 \oplus \cdots \oplus \mathcal{I}_n''y_n)^{\mathcal{J}}$ carries a factor of $N(\mathcal{P})^{k(r_0-r_2)}N(\mathcal{I}_1 \cdots \mathcal{I}_n)^k N(\mathcal{J})^{nk/2}$. Hence, contributing to $c(\Lambda^{\mathcal{J}})$ we have

$$N(\mathcal{P})^{k(r_2-r_0+j)+r_0(r_0+m_1+1)+r_1(r_1+1)/2-j(n+1)}\chi(\mathcal{P})^{2r_2-r_0}\sum_{\Lambda_1} c(\Omega^{\mathcal{J}})$$

where $\Lambda_1^{\mathcal{J}}$ varies over all totally isotropic codimension $n-j$ sublattices of $\Lambda^{\mathcal{J}} \cap \Omega^{\mathcal{J}}/\mathcal{P}(\Lambda^{\mathcal{J}} + \Omega^{\mathcal{J}})$. Summing over all Ω, $\mathcal{P}\Lambda \subseteq \Omega \subseteq \mathcal{P}^{-1}\Lambda$, yields (1).

The proof of (2) is quite similar to the proof of (1), but much simpler, and so we leave it to the reader. □

Remark. As discussed above Proposition 2.2, $\Lambda_1^{\mathcal{J}}/\mathcal{P}\Lambda_1^{\mathcal{J}}$ is a quadratic space over $\mathcal{O}/\mathcal{P}$. By [6, Sec. 42] (for results about quadratic spaces over fields of characteristic 2, see, for example, [9, Sec. 5]), $\Lambda_1^{\mathcal{J}}/\mathcal{P}\Lambda_1^{\mathcal{J}} = R \perp W \perp \mathbb{H}^t$ where $R = \operatorname{rad}\Lambda_1^{\mathcal{J}}/\mathcal{P}\Lambda_1^{\mathcal{J}}$, W is anisotropic, and $\mathbb{H} \simeq \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ denotes a hyperbolic plane. With $U = R \perp W$, [8, Lemma 1.6] and [9, Lemma 4.1] tell us that the number of ℓ-dimensional totally isotropic subspaces of $\Lambda_1^{\mathcal{J}}/\mathcal{P}\Lambda_1^{\mathcal{J}}$ is

$$\varphi_\ell(\Lambda_1^{\mathcal{J}}/\mathcal{P}\Lambda_1^{\mathcal{J}}) = \sum_a q^{(t-a)(\ell-a)}\delta(d+t-\ell+a+1, a)\beta(t,a)\varphi_{\ell-a}(U)$$

where $q = N(\mathcal{P})$, $d = \dim U$, $\delta(m,r) = \prod_{i=0}^{r-1}(q^{m-i}+1)$, $\beta(m,r) = \prod_{i=0}^{r-1}(q^{m-i}-1)/(q^{r-i}-1)$, and $0 \le a \le \ell$. (Note that [8, Lemma 1.6] is proved for a quadratic space over $\mathbb{Z}/p\mathbb{Z}$, but the argument is valid over all finite fields. When the characteristic is 2, we replace Q by $\frac{1}{2}Q$; we present a full discussion of this case in

[9, Sec. 5].) Also, since $U = R \perp W$ with R totally isotropic and W anisotropic, any totally isotropic subspace of U is a subspace of R. Thus $\varphi_{\ell-a}(U) = \beta(r, \ell - a)$ where $r = \dim R$. (So $d = \dim U = r + \dim W$, and since W is anisotropic, $\dim W$ is 0, 1, or 2 by [6, 62:1b] for q odd, and by, e.g., [9, Proposition 5.1] for q even.) Hence

$$\alpha_j(\Omega, \Lambda) = \sum_a q^{(t-a)(\ell-a)} \delta(d+t+a+1, a)\beta(t, a)\beta(r, \ell - a)$$

where $0 \le a \le \ell$, $\ell = r_1 - n + j$.

6. Lemma on Symmetric Coprime Pairs

Definition. We say a pair of matrices (C, D) is a symmetric coprime lower pair for $\Gamma(\mathcal{I}_1, \dots, \mathcal{I}_n; \mathcal{J})$ if:

(a) $C^t D$ is symmetric;
(b) $C \in \mathcal{J}^{-1}\partial\left(\mathcal{I}_i^{-1}\mathcal{I}_j^{-1}\right)$, $D \in \left(\mathcal{I}_i^{-1}\mathcal{I}_j\right)$;
(c) for all prime ideals $\mathcal{P}$,

$$\operatorname{rank}_{\mathcal{P}} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} (C, D) \begin{pmatrix} \alpha\lambda_1 & & \\ & \ddots & \\ & & \alpha\lambda_n \\ & \alpha\lambda_1^{-1} & \\ & \ddots & \\ & & \alpha\lambda_n^{-1} \end{pmatrix} = n$$

where $\alpha\mathcal{O}_{\mathcal{P}} = \mathcal{J}\partial^{-1}\mathcal{O}_{\mathcal{P}}$, $\lambda_i\mathcal{O}_{\mathcal{P}} = \mathcal{I}_i\mathcal{O}_{\mathcal{P}}$.

Since $\Gamma(\mathcal{I}_1, \dots, \mathcal{I}_n; \mathcal{J}) \approx \Gamma(\mathcal{O}, \dots, \mathcal{O}, \mathcal{I}; \mathcal{J})$ when $\mathcal{I} \in \operatorname{cls}(\mathcal{I}_1 \cdots \mathcal{I}_n)$, the following technical lemma focuses on $\Gamma(\mathcal{O}, \dots, \mathcal{O}, \mathcal{I}; \mathcal{J})$, but the conclusion (c) is valid for $\Gamma(\mathcal{I}_1, \dots, \mathcal{I}_n; \mathcal{J})$.

Corresponding definitions and results hold for symmetric coprime upper, right-hand, and left-hand pairs for $\Gamma(\mathcal{I}_1, \dots, \mathcal{I}_n; \mathcal{J})$.

Lemma 6.1. *Let $\mathcal{I}, \mathcal{J}$ be fractional ideals; set*

$$\mathcal{G} = \left\{ E \in \begin{pmatrix} I & \\ & \mathcal{I} \end{pmatrix} \mathcal{O}^{n,n} \begin{pmatrix} I & \\ & \mathcal{I}^{-1} \end{pmatrix} : \ \det E = 1 \right\}.$$

Say $C \in \mathcal{J}^{-1}\partial\left(\begin{smallmatrix} I & \\ & \mathcal{I}^{-1} \end{smallmatrix}\right)\mathcal{O}^{n,n}\left(\begin{smallmatrix} I & \\ & \mathcal{I}^{-1} \end{smallmatrix}\right)$, $D \in \left(\begin{smallmatrix} I & \\ & \mathcal{I}^{-1} \end{smallmatrix}\right)\mathcal{O}^{n,n}\left(\begin{smallmatrix} I & \\ & \mathcal{I} \end{smallmatrix}\right)$ are a symmetric coprime lower pair for $\Gamma(\mathcal{O}, \dots, \mathcal{O}, \mathcal{I}; \mathcal{J})$.

(a) *Let $\mathcal{Q}$ be a prime ideal, and choose $\lambda \in \mathcal{I}$, $\alpha \in \mathcal{J}\partial^{-1}$ such that $\operatorname{ord}_{\mathcal{Q}} \lambda = \operatorname{ord}_{\mathcal{Q}} \mathcal{I}$, $\operatorname{ord}_{\mathcal{Q}} \alpha = \operatorname{ord}_{\mathcal{Q}} \mathcal{J}\partial^{-1}$. Then there is some $E \in \mathcal{G}$ so that $\alpha\left(\begin{smallmatrix} I & \\ & \lambda \end{smallmatrix}\right)CE\left(\begin{smallmatrix} I & \\ & \lambda \end{smallmatrix}\right) \equiv (C_0, 0) \pmod{\mathcal{Q}}$ where C_0 is $n \times r$ with $\operatorname{rank}_{\mathcal{Q}} C_0 = r$.*
(b) *There is a matrix $M \in \Gamma(\mathcal{O}, \dots, \mathcal{O}, \mathcal{I}; \mathcal{J})$ so that $(C|D)M = (C'|D')$ with $\det C', \det D' \neq 0$ and $\left((\mathcal{J}\partial^{-1})^n\mathcal{I}^2 \det C', \det D'\right) = 1$.*
(c) *There are $n \times n$ matrices A, B so that $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma(\mathcal{O}, \dots, \mathcal{O}, \mathcal{I}; \mathcal{J})$.*

Proof. It suffices to establish the claims for $\Gamma(\mathcal{O},\ldots,\mathcal{O},\mathcal{P};\mathcal{J})$ where $\mathcal{P}$ is a prime ideal in $\operatorname{cls}\mathcal{I}$. Choose $\alpha \in \mathcal{J}\partial^{-1}$ so that $\operatorname{ord}_{\mathcal{Q}}\alpha = \operatorname{ord}_{\mathcal{Q}}\mathcal{J}\partial^{-1}$ whenever $\operatorname{ord}_{\mathcal{Q}}\mathcal{J}\partial^{-1} \neq 0$. Choose $\lambda \in \mathcal{P}-\mathcal{P}^2$, $\mu \in \mathcal{P}^{-1}-\mathcal{O}$ so that $\lambda\mu \equiv 1 \pmod{\mathcal{P}}$.

(a) Let $\widetilde{C} = \alpha\left(\begin{smallmatrix} I & \\ & \lambda\end{smallmatrix}\right)C\left(\begin{smallmatrix} I & \\ & \lambda\end{smallmatrix}\right) = (\underline{c}_1\cdots\underline{c}_n)$ (so $\underline{c}_i$ is the ith column of $\widetilde{C}$). Then there is some $E' \in GL_{n-1}(\mathcal{O})$ so that

$$(\underline{c}_1\cdots\underline{c}_{n-1})E \equiv (C_0',0) \pmod{\mathcal{P}}$$

where C_0' is $n\times r'$, $r' = \operatorname{rank}_{\mathcal{P}}(\underline{c}_1\cdots\underline{c}_{n-1})$. So $E_1 = \left(\begin{smallmatrix} E' & \\ & 1\end{smallmatrix}\right) \in \mathcal{G}$, and

$$\alpha\begin{pmatrix} I & \\ & \lambda\end{pmatrix} CE_1 \begin{pmatrix} I & \\ & \lambda\end{pmatrix} = \widetilde{C}E_1 \equiv (C_0',0,\underline{c}_n) \pmod{\mathcal{P}}.$$

First suppose $\underline{c}_n \in \operatorname{span}_{\mathcal{P}} C_0'$. Thus there exist $\gamma_1,\ldots,\gamma_r \in \mathcal{O}$ so that $C_0'\underline{\gamma} \equiv -\underline{c}_n \pmod{\mathcal{P}}$ where $\underline{\gamma} = \left(\begin{smallmatrix}\gamma_1\\ \vdots \\ \gamma_r\end{smallmatrix}\right)$. Let $E = E_1\left(\begin{smallmatrix} I & & \\ & I & \mu\underline{\gamma} \\ & & 1\end{smallmatrix}\right)$, Then

$$\begin{aligned}\alpha\begin{pmatrix} I & \\ & \lambda\end{pmatrix} CE \begin{pmatrix} I & \\ & \lambda\end{pmatrix} &= \widetilde{C}E_1\begin{pmatrix} I & \\ & (\mu\lambda)^{-1}\end{pmatrix}\begin{pmatrix} I & & \underline{\gamma} \\ & I & \\ & & 1\end{pmatrix}\begin{pmatrix} I & \\ & \mu\lambda\end{pmatrix} \\ &\equiv \widetilde{C}E_1\begin{pmatrix} I & \\ & (\mu\lambda)^{-1}\end{pmatrix}\begin{pmatrix} I & & \underline{\gamma} \\ & I & \\ & & 1\end{pmatrix}\begin{pmatrix} I & \\ & \mu\lambda\end{pmatrix} \pmod{\mathcal{P}} \\ &\equiv (C_0',0) \pmod{\mathcal{P}},\end{aligned}$$

proving (a) in the case $\underline{c}_n \in \operatorname{span}_{\mathcal{P}} C_0'$.

So suppose $\underline{c}_n \notin \operatorname{span}_{\mathcal{P}} C_0'$. If $r' = n-1$ then we are done, as then $\operatorname{rank}_{\mathcal{P}}\widetilde{C} = n$ and we can take $E = I$. So let us also suppose $r' < n-1$. Choose $\delta \in \mathcal{P}$ so that for all primes $\mathcal{Q}|\mu\lambda$, $\mathcal{Q} \nmid \delta$. Thus $(\mu\lambda,\delta) = 1$, so there are $u,v \in \mathcal{O}$ so that $v\delta - u\mu\lambda = 1$. Thus $E_2 = \left(\begin{smallmatrix} I & & \\ & \nu & \mu\eta \\ & \lambda & \delta\end{smallmatrix}\right) \in \mathcal{G}$, and

$$\alpha\begin{pmatrix} I & \\ & \lambda\end{pmatrix} CE_1E_2 \begin{pmatrix} I & \\ & \lambda\end{pmatrix} \equiv \widetilde{C}E_1\cdot\begin{pmatrix} I & \\ & \mu\end{pmatrix}E_2\begin{pmatrix} I & \\ & \lambda\end{pmatrix} \pmod{\mathcal{P}}$$

where $\underline{c}_i' = \underline{c}_i$ for $i < n-1$, $\underline{c}_{n-1}' \equiv \underline{c}_n \pmod{\mathcal{P}}$ and $\underline{c}_n' \equiv 0 \pmod{\mathcal{P}}$.

Let $E_3 = \left(\begin{smallmatrix} * & \\ & 1\end{smallmatrix}\right)$ be a permutation matrix that interchanges columns r' and $n-1$. So $E = E_1E_2E_3 \in \mathcal{G}$, and (a) is proved.

(b) Let $\mathcal{A} = (\mathcal{J}\partial^{-1})^n\cdot\mathcal{I}^2\cdot\det C \subseteq \mathcal{O}$. Take $\lambda \in \mathcal{I}$, $\eta \in \mathcal{I}^{-1}$, $\alpha \in \mathcal{J}\partial^{-1}$, $\mu \in \mathcal{J}^{-1}\partial$ so that $\lambda\eta \equiv \alpha\nu \equiv 1 \pmod{\mathcal{Q}}$.

Case 1. Say $\det D \neq 0$, $\det D \notin \mathcal{O}^\times$. Part (a) says there exists some $E \in \mathcal{G}$ so that

$$\alpha\begin{pmatrix} I & \\ & \lambda\end{pmatrix} CE \begin{pmatrix} I & \\ & \lambda\end{pmatrix} \equiv (C_0,0) \pmod{\mathcal{Q}},$$

with C_0 $n\times r$, $\operatorname{rank}_{\mathcal{Q}} C_0 = r$. Thus $\begin{pmatrix} E & \\ & {}^tE^{-1}\end{pmatrix} \in \Gamma(\mathcal{O},\dots,\mathcal{O},\mathcal{I};\mathcal{J})$, and

$$\begin{pmatrix} I & \\ & \lambda\end{pmatrix}(C|D)\begin{pmatrix} E & \\ & {}^tE^{-1}\end{pmatrix}\begin{pmatrix}\alpha I &&& \\ & \alpha\lambda && \\ && I & \\ &&& \eta\end{pmatrix} \equiv (C_0,0)D_0,D_1) \pmod{\mathcal{Q}};$$

here $D_0 \subseteq \operatorname{span}_{\mathcal{Q}} C_0$ (since C^tD is symmetric; see the proof of [5, Lemma 7.2]), and thus $\operatorname{rank}_{\mathcal{Q}}(C_0|D_1) = n$.

Set $W = \mu\begin{pmatrix} I & \\ & \eta\end{pmatrix}\begin{pmatrix} 0_r & \\ & I_{n-r}\end{pmatrix}\begin{pmatrix} I & \\ & \eta\end{pmatrix}$; hence

$$M_{\mathcal{Q}} = \begin{pmatrix} E & \\ & {}^tE^{-1}\end{pmatrix}\begin{pmatrix} I & 0\\ W & I\end{pmatrix} \in \Gamma(\mathcal{O},\dots,\mathcal{O},\mathcal{I};\mathcal{J}).$$

Also, with $(C'|D') = (C|D)M_{\mathcal{Q}}$, $\det D'' = \det D$ and

$$\begin{aligned}&\begin{pmatrix} I & \\ & \lambda\end{pmatrix}(C'|D')\begin{pmatrix}\alpha I &&& \\ & \alpha\lambda && \\ && I & \\ &&& \eta\end{pmatrix}\\ &\equiv \begin{pmatrix} I & \\ & \lambda\end{pmatrix}(C|D)\begin{pmatrix} E & \\ & {}^tE^{-1}\end{pmatrix}\begin{pmatrix}\alpha I &&& \\ & \alpha\lambda && \\ && I & \\ &&& \eta\end{pmatrix}\begin{pmatrix} I &&& \\ & I && \\ 0_r && I & \\ & I && I\end{pmatrix}\\ &\qquad \pmod{\mathcal{Q}}.\end{aligned}$$

Thus $\operatorname{rank}_{\mathcal{Q}} \alpha\begin{pmatrix} I & \\ & \lambda\end{pmatrix}C'\begin{pmatrix} I & \\ & \lambda\end{pmatrix} = n$, and hence $\mathcal{Q} \nmid (\mathcal{J}\partial^{-1})^n\mathcal{I}^2 \det C'$.

Only finitely many prime ideals $\mathcal{Q}$ divide $\det D$, so repeating this process for all such $\mathcal{Q}$ yields a pair $(C'|D') = (C|D)M$, $M \in \Gamma(\mathcal{O},\dots,\mathcal{O},\mathcal{I};\mathcal{J})$, $\det C', \det D' \neq 0$, and $\big((\mathcal{J}\partial^{-1})^n\mathcal{I}^2\det C', \det D'\big) = 1$.

Case 2. Say $\det D \in \mathcal{O}^\times$. If $\det C \neq 0$ then we are done; so suppose $\det C = 0$. Then let $\mathcal{Q}$ be any prime ideal; following the algorithm in Case 1, we produce $M_{\mathcal{Q}} \in \Gamma(\mathcal{O},\dots,\mathcal{O},\mathcal{I};\mathcal{J})$ so that with $(C'|D') = (C|D)M_{\mathcal{Q}}$, we have $\det D' = \det D$ and $\mathcal{Q} \nmid \det C'$ (so $\det C' \neq 0$). This completes this case.

Case 3. Say $\det D = 0$. Choose a prime $\mathcal{Q}$; with λ,η,α,μ as in Case 1, we know there is some $E \in \mathcal{G}$ so that

$$\begin{pmatrix} I & \\ & \lambda\end{pmatrix}(C|D)\begin{pmatrix} E & \\ & {}^tE^{-1}\end{pmatrix}\begin{pmatrix}\alpha I &&& \\ & \alpha\lambda && \\ && I & \\ &&& \eta\end{pmatrix} \equiv (C_0,0|D_0,D_1) \pmod{\mathcal{Q}}$$

with $\operatorname{rank}_{\mathcal{Q}}(C_0|D_1) = n$, $D_0 \subseteq \operatorname{span}_{\mathcal{Q}} C_0$. Thus for some $Y \in \mathcal{O}^{r,r}$, $\operatorname{rank}_{\mathcal{Q}}(C_0Y + D_0, D_1) = n$. Note that Y is uniquely determined modulo $\mathcal{Q}$, and that $C_0{}^tD_0$ is symmetric modulo $\mathcal{Q}$. Since $C_0Y{}^tC_0 \equiv -D_0{}^tC_0 \pmod{\mathcal{Q}}$, we can choose Y to be symmetric in $\mathcal{O}^{r,r}$.

Set

$$Y' = \alpha \begin{pmatrix} I & \\ & \lambda \end{pmatrix} \begin{pmatrix} Y & \\ & 0 \end{pmatrix} \begin{pmatrix} I & \\ & \lambda \end{pmatrix};$$

hence $M = \left(\begin{smallmatrix} E & \\ & {}^tE^{-1} \end{smallmatrix}\right)\left(\begin{smallmatrix} I & Y' \\ & I \end{smallmatrix}\right) \in \Gamma(\mathcal{O}, \ldots, \mathcal{O}, \mathcal{I}; \mathcal{J})$. Thus with $(C'|D') = (C|D)M$, we have

$$\begin{pmatrix} I & \\ & \lambda \end{pmatrix} (C'|D') \begin{pmatrix} \alpha I & & & \\ & \alpha\lambda & & \\ & & I & \\ & & & \eta \end{pmatrix} \equiv (C_0, 0|C_0Y + D_0, D_1) \pmod{\mathcal{Q}}.$$

Hence $\operatorname{rank}_{\mathcal{Q}}\left(\begin{smallmatrix} I & \\ & \lambda \end{smallmatrix}\right)D'\left(\begin{smallmatrix} I & \\ & \eta \end{smallmatrix}\right) = n$, so $\mathcal{Q} \nmid \det D'$ (and thus $\det D' \neq 0$). This reduces this situation to one of the previous cases.

(c) Set $\kappa = \det D$; choose $\lambda \in \mathcal{A}$ so that $(\kappa, \lambda) = 1$, and $\eta \in \mathcal{O}$ so that $\eta \equiv \kappa^{-1} \pmod{\lambda}$ (this is possible by the Chinese Remainder Theorem). Since $C \in \mathcal{J}^{-1}\partial\left(\begin{smallmatrix} I & \\ & \mathcal{I}^{-1} \end{smallmatrix}\right)\mathcal{O}^{n,n}\left(\begin{smallmatrix} I & \\ & \mathcal{I}^{-1} \end{smallmatrix}\right)$, we have $\lambda{}^tC^{-1} \in \mathcal{J}\partial^{-1}\left(\begin{smallmatrix} I & \\ & \mathcal{I} \end{smallmatrix}\right)\mathcal{O}^{n,n}\left(\begin{smallmatrix} I & \\ & \mathcal{I} \end{smallmatrix}\right)$. Also, $\lambda|(\eta\kappa - 1)$, so

$$B = (\eta\kappa - 1){}^tC^{-1} \in \mathcal{J}\partial^{-1} \begin{pmatrix} I & \\ & \mathcal{I} \end{pmatrix} \mathcal{O}^{n,n} \begin{pmatrix} I & \\ & \mathcal{I} \end{pmatrix}.$$

(Note that locally everywhere: $C \in \mathcal{J}^{-1}\partial\left(\begin{smallmatrix} I & \\ & \mathcal{I}^{-1} \end{smallmatrix}\right)C_0\left(\begin{smallmatrix} I & \\ & \mathcal{I}^{-1} \end{smallmatrix}\right)$, $C_0 \in \mathcal{O}^{n,n}$, $\det C_0 = (\mathcal{J}\partial^{-1})^n\mathcal{I}^2 \det C$; hence $C^{-1} \in \frac{1}{\det C_0}\mathcal{J}\partial^{-1}\left(\begin{smallmatrix} I & \\ & \mathcal{I} \end{smallmatrix}\right)\mathcal{O}^{n,n}\left(\begin{smallmatrix} I & \\ & \mathcal{I} \end{smallmatrix}\right)$.)

Set $A = \eta\kappa{}^tD^{-1}$. We have $\kappa = \det D$, $\eta \in \mathcal{O}$, so $\eta\kappa{}^tD^{-1} \in \left(\begin{smallmatrix} I & \\ & \mathcal{I} \end{smallmatrix}\right)\mathcal{O}^{n,n}\left(\begin{smallmatrix} I & \\ & \mathcal{I}^{-1} \end{smallmatrix}\right)$. Thus $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is a candidate for $\Gamma(\mathcal{O}, \ldots, \mathcal{O}, \mathcal{I}; \mathcal{J})$. To see this matrix indeed lies in this group, first note that $A{}^tD - B{}^tC = I$, and by assumption, $C{}^tD$ is symmetric. Finally, substituting for A and B, we get $A{}^tB = \eta\kappa(\eta\kappa - 1){}^tD^{-1}C^{-1}$; since ${}^tD^{-1}C^{-1} = (C{}^tD)^{-1}$ is symmetric, so is $A{}^tB$. $\square$

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